2022 AMC 12A

Time limit: 75 minutes

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https://live.poshenloh.com/past-contests/amc12/2022A

1. What is the value of

$$3+\frac{1}{3+\frac{1}{3+\frac{1}{3}}}?$$

A $\frac{31}{10}$

B $\frac{49}{15}$

C $\frac{33}{10}$

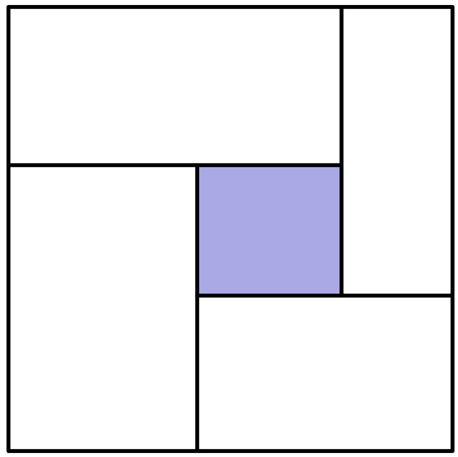
D $\frac{109}{33}$

E $\frac{15}{4}$

2. The sum of three numbers is $96.$ The first number is 6 times the third number, and the third number is 40 less than the second number. What is the absolute value of the difference between the first and second numbers?

- A 1
- B 2
- C 3
- D 4
- E 5

3. Five rectangles, $A, B, C, D,$ and $E,$ are arranged in a square as shown below. These rectangles have dimensions $1 \times 6, 2 \times 4, 5 \times 6, 2 \times 7,$ and $2 \times 3,$ respectively. (The figure is not drawn to scale.) Which of the five rectangles is the shaded one in the middle?



A A

B B

C C

D D

E E

4. The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is the sum of the digits of n?

A 3

B 6

C 8

D 9

E 12

5. Let the *taxicab distance* between points (x_1, y_1) and (x_2, y_2) in the coordinate plane be given by $|x_1 - x_2| + |y_1 - y_2|$. For how many points P with integer coordinates is the taxicab distance between P and the origin less than or equal to 20?

A 441

B 761

C 841

D 921

E 924

6. A data set consists of 6 (not distinct) positive integers: $1, 7, 5, 2, 5,$ and X. The average (arithmetic mean) of the 6 numbers equals a value in the data set. What is the sum of all positive values of X?

A 10

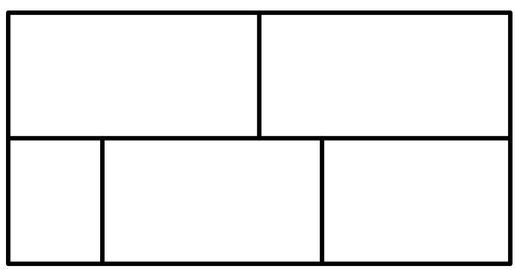
B 26

C 32

D 36

E 40

7. A rectangle is partitioned into 5 regions as shown. Each region is to be painted a solid color - red, orange, yellow, blue, or green - so that regions that touch are painted different colors, and colors can be used more than once. How many different colorings are possible?



A 120

B 270

C 360

D 540

E 720

8. The infinite product

$$\sqrt[3]{10} \cdot \sqrt[3]{\sqrt[3]{10}} \cdot \sqrt[3]{\sqrt[3]{\sqrt[3]{10}}} \cdots$$

evaluates to a real number. What is that number?

A $\sqrt{10}$

B $\sqrt[3]{100}$

C $\sqrt[4]{1000}$

D 10

E $10\sqrt[3]{10}$

9. On Halloween 31 children walked into the principal's office asking for candy. They can be classified into three types: some always lie; some always tell the truth; and some alternately lie and tell the truth. The alternaters arbitrarily choose their first response, either a lie or the truth, but each subsequent statement has the opposite truth value from its predecessor. The principal asked everyone the same three questions in this order.

"Are you a truth-teller?" The principal gave a piece of candy to each of the 22 children who answered yes.

"Are you an alternater?" The principal gave a piece of candy to each of the 15 children who answered yes.

"Are you a liar?" The principal gave a piece of candy to each of the 9 children who answered yes.

How many pieces of candy in all did the principal give to the children who always tell the truth?

A 7

B 12

C 21

D 27

E 31

10. What is the number of ways the numbers from 1 to 14 can be split into 7 pairs such that for each pair, the greater number is at least 2 times the smaller number?

A 108

B 120

C 126

D 132

E 144

11. What is the product of all real numbers x such that the distance on the number line between $\log_6 x$ and $\log_6 9$ is twice the distance on the number line between $\log_6 10$ and 1?

A 10

B 18

C 25

D 36

E 81

12. Let M be the midpoint of $\overline{AB}$ in regular tetrahedron $ABCD$. What is $\cos(\angle CMD)$?

A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{2}{5}$

D $\frac{1}{2}$

E $\frac{\sqrt{3}}{2}$

13. Let $\mathcal{R}$ be the region in the complex plane consisting of all complex numbers z that can be written as the sum of complex numbers z_1 and z_2, where z_1 lies on the segment with endpoints 3 and $4i$, and z_2 has magnitude at most 1. What integer is closest to the area of $\mathcal{R}$?

A 13

B 14

C 15

D 16

E 17

14. What is the value of

$$(\log 5)^3 + (\log 20)^3 + (\log 8)(\log 0.25)$$

where $\log$ denotes the base-ten logarithm?

- A $\frac{3}{2}$
- B $\frac{7}{4}$
- C 2
- D $\frac{9}{4}$
- E 3

15. The roots of the polynomial $10x^3 - 39x^2 + 29x - 6$ are the height, length, and width of a rectangular box (right rectangular prism). A new rectangular box is formed by lengthening each edge of the original box by 2 units. What is the volume of the new box?

- A $\frac{24}{5}$
- B $\frac{42}{5}$
- C $\frac{81}{5}$
- D 30
- E 48

16. A *triangular number* is a positive integer that can be expressed in the form $t_n = 1 + 2 + 3 + \cdots + n$, for some positive integer n. The three smallest triangular numbers that are also perfect squares are $t_1 = 1 = 1^2$, $t_8 = 36 = 6^2$, and $t_{49} = 1225 = 35^2$. What is the sum of the digits of the fourth smallest triangular number that is also a perfect square?

- A 6
- B 9
- C 12
- D 18
- E 27

17. Suppose a is a real number such that the equation

$$a \cdot (\sin x + \sin(2x)) = \sin(3x)$$

has more than one solution in the interval $(0, \pi)$. The set of all such a can be written in the form $(p, q) \cup (q, r)$, where p, q, and r are real numbers with $p < q < r$. What is $p + q + r$?

- A -4
- B -1
- C 0
- D 1
- E 4

18. Let T_k be the transformation of the coordinate plane that first rotates the plane k degrees counterclockwise around the origin and then reflects the plane across the y-axis. What is the least positive integer n such that performing the sequence of transformations $T_1, T_2, T_3, \ldots, T_n$ returns the point $(1, 0)$ back to itself?

- A 359
- B 360
- C 719
- D 720
- E 721

19. Suppose that 13 cards numbered $1, 2, 3, \ldots, 13$ are arranged in a row. The task is to pick them up in numerically increasing order, working repeatedly from left to right. In the example below, cards $1, 2, 3$ are picked up on the first pass, 4 and 5 on the second pass, 6 on the third pass, $7, 8, 9, 10$ on the fourth pass, and $11, 12, 13$ on the fifth pass. For how many of the $13!$ possible orderings of the cards will the 13 cards be picked up in exactly two passes?

7	11	8	6	4	5	9	12	1	13	10	2	3

- A 4082
- B 4095
- C 4096
- D 8178
- E 8191

20. Isosceles trapezoid $ABCD$ has parallel sides $\overline{AD}$ and $\overline{BC}$, with $BC < AD$ and $AB = CD$. There is a point P in the plane such that $PA = 1, PB = 2, PC = 3,$ and $PD = 4$. What is $\frac{BC}{AD}$?

A $\frac{1}{4}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{3}{4}$

21. Let $P(x) = x^{2022} + x^{1011} + 1$. Which of the following polynomials divides $P(x)$?

A $x^2 - x + 1$

B $x^2 + x + 1$

C $x^4 + 1$

D $x^6 - x^3 + 1$

E $x^6 + x^3 + 1$

22. Let c be a real number, and let z_1, z_2 be the two complex numbers satisfying the quadratic $z^2 - cz + 10 = 0$. Points $z_1, z_2, \frac{1}{z_1}$, and $\frac{1}{z_2}$ are the vertices of a (convex) quadrilateral Q in the complex plane. When the area of Q obtains its maximum value, c is the closest to which of the following?

A 4.5

B 5

C 5.5

D 6

E 6.5

23. Let h_n and k_n be the unique relatively prime positive integers such that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{h_n}{k_n}.$$

Let L_n denote the least common multiple of the numbers $1, 2, 3, \ldots, n$. For how many integers n with $1 \le n \le 22$ is $k_n < L_n$?

A 0

B 3

C 7

D 8

E 10

24. How many strings of length 5 formed from the digits $0, 1, 2, 3, 4$ are there such that for each $j \in \{1, 2, 3, 4\}$, at least j of the digits are less than j? (For example, 02214 satisfies the condition because it contains at least 1 digit less than 1, at least 2 digits less than 2, at least 3 digits less than 3, and at least 4 digits less than 4. The string 23404 does not satisfy the condition because it does not contain at least 2 digits less than 2.)

- A 500
- B 625
- C 1089
- D 1199
- E 1296

25. A circle with integer radius r is centered at (r, r). Distinct line segments of length c_i connect points $(0, a_i)$ to $(b_i, 0)$ for $1 \leq i \leq 14$ and are tangent to the circle, where a_i, b_i, and c_i are all positive integers and $c_1 \leq c_2 \leq \cdots \leq c_{14}$. What is the ratio $\frac{c_{14}}{c_1}$ for the least possible value of r?

- A $\frac{21}{5}$
- B $\frac{85}{13}$
- C 7
- D $\frac{39}{5}$
- E 17

Solutions: https://live.poshenloh.com/past-contests/amc12/2022A/solutions