

# 2019 AIME II

Time limit: 180 minutes

Typeset by: LIVE by Po-Shen Loh

<https://live.poshenloh.com/past-contests/aime/2019II>



Copyright: Mathematical Association of America. Reproduced with permission.

1. Two different points,  $C$  and  $D$ , lie on the same side of line  $AB$  so that  $\triangle ABC$  and  $\triangle BAD$  are congruent with  $AB = 9$ ,  $BC = AD = 10$ , and  $CA = DB = 17$ . The intersection of these two triangular regions has area  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
  
2. Lily pads  $1, 2, 3, \dots$  lie in a row on a pond. A frog makes a sequence of jumps starting on pad 1. From any pad  $k$  the frog jumps to either pad  $k + 1$  or pad  $k + 2$  chosen randomly with probability  $\frac{1}{2}$  and independently of other jumps. The probability that the frog visits pad 7 is  $\frac{p}{q}$ , where  $p$  and  $q$  are relatively prime positive integers. Find  $p + q$ .

3. Find the number of 7-tuples of positive integers  $(a, b, c, d, e, f, g)$  that satisfy the following system of equations:

$$abc = 70,$$

$$cde = 71,$$

$$efg = 72.$$

4. A standard six-sided fair die is rolled four times. The probability that the product of all four numbers rolled is a perfect square is  $\frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
5. Four ambassadors and one advisor for each of them are to be seated at a round table with 12 chairs numbered in order 1 to 12. Each ambassador must sit in an even-numbered chair. Each advisor must sit in a chair adjacent to his or her ambassador. There are  $N$  ways for the 8 people to be seated at the table under these conditions. Find the remainder when  $N$  is divided by 1000.

6. In a Martian civilization, all logarithms whose bases are not specified are assumed to be base  $b$ , for some fixed  $b \geq 2$ . A Martian student writes down

$$3 \log(\sqrt{x} \log x) = 56$$

$$\log_{\log x}(x) = 54$$

and finds that this system of equations has a single real number solution  $x > 1$ . Find  $b$ .

7. Triangle  $ABC$  has side lengths  $AB = 120$ ,  $BC = 220$ , and  $AC = 180$ . Lines  $\ell_A$ ,  $\ell_B$ , and  $\ell_C$  are drawn parallel to  $\overline{BC}$ ,  $\overline{AC}$ , and  $\overline{AB}$ , respectively, such that the intersections of  $\ell_A$ ,  $\ell_B$ , and  $\ell_C$  with the interior of  $\triangle ABC$  are segments of lengths 55, 45, and 15, respectively. Find the perimeter of the triangle whose sides lie on lines  $\ell_A$ ,  $\ell_B$ , and  $\ell_C$ .
8. The polynomial  $f(z) = az^{2018} + bz^{2017} + cz^{2016}$  has real coefficients not exceeding 2019, and  $f\left(\frac{1+\sqrt{3}i}{2}\right) = 2015 + 2019\sqrt{3}i$ . Find the remainder when  $f(1)$  is divided by 1000.

9. Call a positive integer  $n$   $k$ -pretty if  $n$  has exactly  $k$  positive divisors and  $n$  is divisible by  $k$ . For example, 18 is 6-pretty. Let  $S$  be the sum of the positive integers less than 2019 that are 20-pretty. Find  $\frac{S}{20}$ .
10. There is a unique angle  $\theta$  between  $0^\circ$  and  $90^\circ$  such that for nonnegative integers  $n$ , the value of  $\tan(2^n\theta)$  is positive when  $n$  is a multiple of 3, and negative otherwise. The degree measure of  $\theta$  is  $\frac{p}{q}$ , where  $p$  and  $q$  are relatively prime positive integers. Find  $p + q$ .
11. Triangle  $ABC$  has side lengths  $AB = 7$ ,  $BC = 8$ , and  $CA = 9$ . Circle  $\omega_1$  passes through  $B$  and is tangent to line  $AC$  at  $A$ . Circle  $\omega_2$  passes through  $C$  and is tangent to line  $AB$  at  $A$ . Let  $K$  be the intersection of circles  $\omega_1$  and  $\omega_2$  not equal to  $A$ . Then  $AK = \frac{m}{n}$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
12. For  $n \geq 1$  call a finite sequence  $(a_1, a_2, \dots, a_n)$  of positive integers *progressive* if  $a_i < a_{i+1}$  and  $a_i$  divides  $a_{i+1}$  for  $1 \leq i \leq n - 1$ . Find the number of progressive sequences such that the sum of the terms in the sequence is equal to 360.

13. Regular octagon  $A_1A_2A_3A_4A_5A_6A_7A_8$  is inscribed in a circle of area 1. Point  $P$  lies inside the circle so that the region bounded by  $\overline{PA_1}$ ,  $\overline{PA_2}$ , and the minor arc  $\widehat{A_1A_2}$  of the circle has area  $\frac{1}{7}$ , while the region bounded by  $\overline{PA_3}$ ,  $\overline{PA_4}$ , and the minor arc  $\widehat{A_3A_4}$  of the circle has area  $\frac{1}{9}$ . There is a positive integer  $n$  such that the area of the region bounded by  $\overline{PA_6}$ ,  $\overline{PA_7}$ , and the minor arc  $\widehat{A_6A_7}$  of the circle is equal to  $\frac{1}{8} - \frac{\sqrt{2}}{n}$ . Find  $n$ .
14. Find the sum of all positive integers  $n$  such that, given an unlimited supply of stamps of denominations 5,  $n$ , and  $n + 1$  cents, 91 cents is the greatest postage that cannot be formed.
15. In acute triangle  $ABC$ , points  $P$  and  $Q$  are the feet of the perpendiculars from  $C$  to  $\overline{AB}$  and from  $B$  to  $\overline{AC}$ , respectively. Line  $PQ$  intersects the circumcircle of  $\triangle ABC$  in two distinct points,  $X$  and  $Y$ . Suppose  $XP = 10$ ,  $PQ = 25$ , and  $QY = 15$ . The value of  $AB \cdot AC$  can be written in the form  $m\sqrt{n}$ , where  $m$  and  $n$  are positive integers, and  $n$  is not divisible by the square of any prime. Find  $m + n$ .

Solutions: <https://live.poshenloh.com/past-contests/aime/2019II/solutions>

