

2010 AIME I

Time limit: 180 minutes

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1. Maya lists all the positive divisors of 2010^2 . She then randomly selects two distinct divisors from this list. Let p be the probability that exactly one of the selected divisors is a perfect square. The probability p can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

2. Find the remainder when

$$9 \cdot 99 \cdot 999 \cdot \dots \cdot \underbrace{99 \dots 9}_{999 \text{ 9's}}$$

is divided by 1000.

3. Suppose that $y = \frac{3}{4}x$ and $x^y = y^x$. The quantity $x + y$ can be expressed as a rational number $\frac{r}{s}$, where r and s are relatively prime positive integers. Find $r + s$.

4. Jackie and Phil have two fair coins and a third coin that comes up heads with probability $\frac{4}{7}$. Jackie flips the three coins, and then Phil flips the three coins. Let $\frac{m}{n}$ be the probability that Jackie gets the same number of heads as Phil, where m and n are relatively prime positive integers. Find $m + n$.

5. Positive integers $a, b, c,$ and d satisfy $a > b > c > d, a + b + c + d = 2010,$ and $a^2 - b^2 + c^2 - d^2 = 2010$. Find the number of possible values of a .

6. Let $P(x)$ be a quadratic polynomial with real coefficients satisfying

$$x^2 - 2x + 2 \leq P(x) \leq 2x^2 - 4x + 3$$

for all real numbers x , and suppose $P(11) = 181$. Find $P(16)$.

7. Define an ordered triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of sets to be *minimally intersecting* if $|\mathcal{A} \cap \mathcal{B}| = |\mathcal{B} \cap \mathcal{C}| = |\mathcal{C} \cap \mathcal{A}| = 1$ and $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C} = \emptyset$. For example, $(\{1, 2\}, \{2, 3\}, \{1, 3, 4\})$ is a minimally intersecting triple. Let N be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. Find the remainder when N is divided by 1000.

Note: $|\mathcal{S}|$ represents the number of elements in the set \mathcal{S} .

8. For a real number a , let $\lfloor a \rfloor$ denote the greatest integer less than or equal to a . Let \mathcal{R} denote the region in the coordinate plane consisting of points (x, y) such that

$$\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 25.$$

The region \mathcal{R} is completely contained in a disk of radius r (a disk is the union of a circle and its interior). The minimum value of r can be written as $\frac{\sqrt{m}}{n}$, where m and n are integers and m is not divisible by the square of any prime. Find $m + n$.

9. Let (a, b, c) be a real solution of the system of equations

$$x^3 - xyz = 2, \quad y^3 - xyz = 6, \quad z^3 - xyz = 20.$$

The greatest possible value of $a^3 + b^3 + c^3$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

10. Let N be the number of ways to write 2010 in the form

$$2010 = a_3 \cdot 10^3 + a_2 \cdot 10^2 + a_1 \cdot 10 + a_0,$$

where the a_i 's are integers, and $0 \leq a_i \leq 99$. An example of such a representation is $1 \cdot 10^3 + 3 \cdot 10^2 + 67 \cdot 10^1 + 40 \cdot 10^0$. Find N .

11. Let \mathcal{R} be the region consisting of the set of points in the coordinate plane that satisfy both $|8 - x| + y \leq 10$ and $3y - x \geq 15$. When \mathcal{R} is revolved around the line whose equation is $3y - x = 15$, the volume of the resulting solid is $\frac{m\pi}{n\sqrt{p}}$, where m , n , and p are positive integers, m and n are relatively prime, and p is not divisible by the square of any prime. Find $m + n + p$.

12. Let $m \geq 3$ be an integer and let $S = \{3, 4, 5, \dots, m\}$. Find the smallest value of m such that for every partition of S into two subsets, at least one of the subsets contains integers a , b , and c (not necessarily distinct) such that $ab = c$.

Note: a partition of S is a pair of sets A, B such that $A \cap B = \emptyset$ and $A \cup B = S$.

13. Rectangle $ABCD$ and a semicircle with diameter \overline{AB} are coplanar and have nonoverlapping interiors. Let \mathcal{R} denote the region enclosed by the semicircle and the rectangle. Line ℓ meets the semicircle, segment \overline{AB} , and segment \overline{CD} at distinct points N , U , and T , respectively. Line ℓ divides region \mathcal{R} into two regions with areas in the ratio $1 : 2$. Suppose that $AU = 84$, $AN = 126$, and $UB = 168$. Then DA can be represented as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

14. For each positive integer n , let $f(n) = \sum_{k=1}^{100} \lfloor \log_{10}(kn) \rfloor$. Find the largest value of n for which $f(n) \leq 300$.

Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x .

15. In $\triangle ABC$ with $AB = 12$, $BC = 13$, and $AC = 15$, let M be a point on \overline{AC} such that the incircles of $\triangle ABM$ and $\triangle BCM$ have equal radii. Let p and q be positive relatively prime integers such that $\frac{AM}{CM} = \frac{p}{q}$. Find $p + q$.

Solutions: <https://live.poshenloh.com/past-contests/aimc/2010I/solutions>

