2009 AIME I

Time limit: 180 minutes

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1. Call a 3-digit number *geometric* if it has 3 distinct digits which, when read from left to right, form a geometric sequence. Find the difference between the largest and smallest geometric numbers.

2. There is a complex number z with imaginary part 164 and a positive integer n such that

$$\frac{z}{z+n} = 4i.$$

Find n.

3. A coin that comes up heads with probability $p > 0$ and tails with probability $1 - p > 0$ independently on each flip is flipped eight times. Suppose the probability of three heads and five tails is equal to $\frac{1}{25}$ of the probability of five heads and three tails. Let $p = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

4. In parallelogram $ABCD$, point M is on $\overline{AB}$ so that $\frac{AM}{AB} = \frac{17}{1000}$, and point N is on $\overline{AD}$ so that $\frac{AN}{AD} = \frac{17}{2009}$. Let P be the point of intersection of $\overline{AC}$ and $\overline{MN}$. Find $\frac{AC}{AP}$.

5. Triangle ABC has $AC = 450$ and $BC = 300$. Points K and L are located on $\overline{AC}$ and $\overline{AB}$ respectively so that $AK = CK$, and $\overline{CL}$ is the angle bisector of angle C. Let P be the point of intersection of $\overline{BK}$ and $\overline{CL}$, and let M be the point on line BK for which K is the midpoint of $\overline{PM}$. If $AM = 180$, find LP.

6. How many positive integers N less than 1000 are there such that the equation $x^{\lfloor x \rfloor} = N$ has a solution for x? (The notation $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.)

7. The sequence (a_n) satisfies $a_1 = 1$ and $5^{(a_{n+1} - a_n)} - 1 = \frac{1}{n + \frac{2}{3}}$ for $n \geq 1$. Let k be the least integer greater than 1 for which a_k is an integer. Find k.

8. Let $S = \{2^0, 2^1, 2^2, \ldots, 2^{10}\}$. Consider all possible positive differences of pairs of elements of S. Let N be the sum of all of these differences. Find the remainder when N is divided by 1000.

9. A game show offers a contestant three prizes A, B and C, each of which is worth a whole number of dollars from $\$1$ to $\$9999$ inclusive. The contestant wins the prizes by correctly guessing the price of each prize in the order A, B, C. As a hint, the digits of the three prices are given. On a particular day, the digits given were $1, 1, 1, 1, 3, 3, 3$. Find the total number of possible guesses for all three prizes consistent with the hint.

10. The Annual Interplanetary Mathematics Examination (AIME) is written by a committee of five Martians, five Venusians, and five Earthlings. At meetings, committee members sit at a round table with chairs numbered from 1 to 15 in clockwise order. Committee rules state that a Martian must occupy chair 1 and an Earthling must occupy chair 15. Furthermore, no Earthling can sit immediately to the left of a Martian, no Martian can sit immediately to the left of a Venusian, and no Venusian can sit immediately to the left of an Earthling. The number of possible seating arrangements for the committee is $N \cdot (5!)^3$. Find N.

11. Consider the set of all triangles OPQ where O is the origin and P and Q are distinct points in the plane with nonnegative integer coordinates (x, y) such that $41x + y = 2009$. Find the number of such distinct triangles whose area is a positive integer.

12. In right $\triangle ABC$ with hypotenuse $\overline{AB}$, $AC = 12$, $BC = 35$, and $\overline{CD}$ is the altitude to $\overline{AB}$. Let ω be the circle having $\overline{CD}$ as a diameter. Let I be a point outside $\triangle ABC$ such that $\overline{AI}$ and $\overline{BI}$ are both tangent to circle ω. The ratio of the perimeter of $\triangle ABI$ to the length AB can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

13. The terms of the sequence (a_i) defined by $a_{n+2} = \frac{a_n + 2009}{1 + a_{n+1}}$ for $n \geq 1$ are positive integers. Find the minimum possible value of $a_1 + a_2$.

14. For $t = 1, 2, 3, 4$, define $S_t = \sum_{i=1}^{350} a_i^t$, where $a_i \in \{1, 2, 3, 4\}$. If $S_1 = 513$ and $S_4 = 4745$, find the minimum possible value for S_2.

15. In triangle ABC, $AB = 10$, $BC = 14$, and $CA = 16$. Let D be a point in the interior of $\overline{BC}$. Let I_B and I_C denote the incenters of triangles ABD and ACD, respectively. The circumcircles of triangles BI_BD and CI_CD meet at distinct points P and D. The maximum possible area of $\triangle BPC$ can be expressed in the form $a - b\sqrt{c}$, where a, b, and c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.

Solutions: https://live.poshenloh.com/past-contests/aime/2009I/solutions