2007 AIME II

Time limit: 180 minutes

Typeset by: LIVE by Po-Shen Loh
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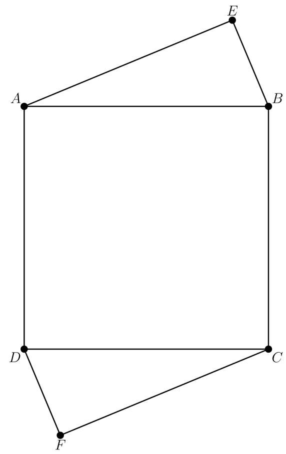
1. A mathematical organization is producing a set of commemorative license plates. Each plate contains a sequence of five characters chosen from the four letters in AIME and the four digits in $2007.$ No character may appear in a sequence more times than it appears among the four letters in AIME or the four digits in $2007.$ A set of plates in which each possible sequence appears exactly once contains N license plates. Find $\frac{N}{10}.$

2. Find the number of ordered triples (a, b, c) where $a, b,$ and c are positive integers, a is a factor of $b,$ a is a factor of $c,$ and $a + b + c = 100.$

3. Square $ABCD$ has side length 13, and points E and F are exterior to the square such that $BE = DF = 5$ and $AE = CF = 12$. Find EF^2.



4. The workers in a factory produce widgets and whoosits. For each product, production time is constant and identical for all workers, but not necessarily equal for the two products. In one hour, 100 workers can produce 300 widgets and 200 whoosits. In two hours, 60 workers can produce 240 widgets and 300 whoosits. In three hours, 50 workers can produce 150 widgets and m whoosits. Find m.

5. The graph of the equation $9x + 223y = 2007$ is drawn on graph paper with each square representing one unit in each direction. How many of the 1 by 1 graph paper squares have interiors lying entirely below the graph and entirely in the first quadrant?

6. An integer is called *parity-monotonic* if its decimal representation $a_1a_2a_3\ldots a_k$ satisfies $a_i < a_{i+1}$ if a_i is odd, and $a_i > a_{i+1}$ if a_i is even. How many four-digit parity-monotonic integers are there?

7. Given a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. For a certain integer k, there are exactly 70 positive integers $n_1, n_2, \ldots, n_{70}$ such that

$$k = \lfloor \sqrt[3]{n_1} \rfloor = \lfloor \sqrt[3]{n_2} \rfloor = \cdots = \lfloor \sqrt[3]{n_{70}} \rfloor$$

and k divides n_i for all i such that $1 \leq i \leq 70$. Find the maximum value of $\frac{n_i}{k}$ for $1 \leq i \leq 70.$

8. A rectangular piece of paper measures 4 units by 5 units. Several lines are drawn parallel to the edges of the paper. A rectangle determined by the intersections of some of these lines is called *basic* if (i) all four sides of the rectangle are segments of drawn line segments, and (ii) no segments of drawn lines lie inside the rectangle.

Given that the total length of all lines drawn is exactly 2007 units, let N be the maximum possible number of basic rectangles determined. Find the remainder when N is divided by $1000.$

9. Rectangle $ABCD$ is given with $AB = 63$ and $BC = 448.$ Points E and F lie on $\overline{AD}$ and $\overline{BC}$ respectively, such that $AE = CF = 84.$ The inscribed circle of triangle BEF is tangent to $\overline{EF}$ at point $P,$ and the inscribed circle of triangle DEF is tangent to $\overline{EF}$ at point $Q.$ Find $PQ.$

10. Let S be a set with six elements. Let $\mathcal{P}$ be the set of all subsets of $S.$ Subsets A and B of $S,$ not necessarily distinct, are chosen independently and at random from $\mathcal{P}.$ The probability that B is contained in at least one of A or $S - A$ is $\frac{m}{n^r},$ where $m, n,$ and r are positive integers, n is prime, and m and n are relatively prime. Find $m + n + r.$ (The set $S - A$ is the set of all elements of S which are not in $A.$)

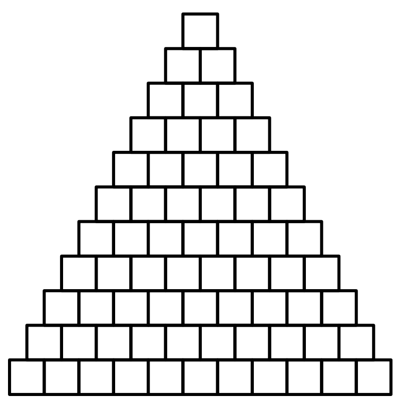
11. Two long cylindrical tubes of the same length but different diameters lie parallel to each other on a flat surface. The larger tube has radius 72 and rolls along the surface toward the smaller tube, which has radius $24.$ It rolls over the smaller tube and continues rolling along the flat surface until it comes to rest on the same point of its circumference as it started, having made one complete revolution. If the smaller tube never moves, and the rolling occurs with no slipping, the larger tube ends up a distance x from where it starts. The distance x can be expressed in the form $a\pi + b\sqrt{c},$ where $a, b,$ and c are integers and c is not divisible by the square of any prime. Find $a + b + c.$

12. The increasing geometric sequence $x_0, x_1, x_2, \ldots$ consists entirely of integral powers of 3. Given that

$$\sum_{n=0}^{7} \log_3(x_n) = 308 \qquad \text{and} \qquad 56 \le \log_3\left(\sum_{n=0}^{7} x_n\right) \le 57,$$

find $\log_3(x_{14})$.

13. A triangular array of squares has one square in the first row, two in the second, and, in general, k squares in the kth row for $1 \le k \le 11$. With the exception of the bottom row, each square rests on two squares in the row immediately below, as illustrated in the figure. In each square of the eleventh row, a 0 or a 1 is placed. Numbers are then placed into the other squares, with the entry for each square being the sum of the entries in the two squares below it. For how many initial distributions of 0's and 1's in the bottom row is the number in the top square a multiple of 3?



14. Let $f(x)$ be a polynomial with real coefficients such that $f(0) = 1, f(2) + f(3) = 125$, and for all x, $f(x)f(2x^2) = f(2x^3 + x)$. Find $f(5)$.

15. Four circles $\omega, \omega_A, \omega_B$, and ω_C with the same radius are drawn in the interior of triangle ABC such that ω_A is tangent to sides AB and AC, ω_B to BC and BA, ω_C to CA and CB, and ω is externally tangent to ω_A, ω_B, and ω_C. If the sides of triangle ABC are $13, 14$, and 15, the radius of ω can be represented in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solutions: https://live.poshenloh.com/past-contests/aime/2007II/solutions