2007 AIME I

Time limit: 180 minutes

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1. How many positive perfect squares less than 10^6 are multiples of 24?

2. A 100 foot long moving walkway moves at a constant rate of 6 feet per second. Al steps onto the start of the walkway and stands. Bob steps onto the start of the walkway two seconds later and strolls forward along the walkway at a constant rate of 4 feet per second. Two seconds after that, Cy reaches the start of the walkway and walks briskly forward beside the walkway at a constant rate of 8 feet per second. At a certain time, one of these three persons is exactly halfway between the other two. At that time, find the distance in feet between the start of the walkway and the middle person.

3. The complex number z is equal to $9 + bi$, where b is a positive real number and $i^2 = -1$. Given that the imaginary parts of z^2 and z^3 are equal, find b.

4. Three planets revolve about a star in coplanar circular orbits with the star at the center. All planets revolve in the same direction, each at a constant speed, and the periods of their orbits are $60, 84,$ and 140 years. The positions of the star and all three planets are currently collinear. They will next be collinear after n years. Find $n.$

5. The formula for converting a Fahrenheit temperature F to the corresponding Celsius temperature C is $C = \frac{5}{9}(F - 32).$ An integer Fahrenheit temperature is converted to Celsius and rounded to the nearest integer; the resulting integer Celsius temperature is converted back to Fahrenheit and rounded to the nearest integer. For how many integer Fahrenheit temperatures T with $32 \leq T \leq 1000$ does the original temperature equal the final temperature?

6. A frog is placed at the origin on the number line, and moves according to the following rule: in a given move, the frog advances to either the closest point with a greater integer coordinate that is a multiple of $3,$ or to the closest point with a greater integer coordinate that is a multiple of $13.$ A *move sequence* is a sequence of coordinates which correspond to valid moves, beginning with $0,$ and ending with $39.$ For example, $0, 3, 6, 13, 15, 26, 39$ is a move sequence. How many move sequences are possible for the frog?

7. Let

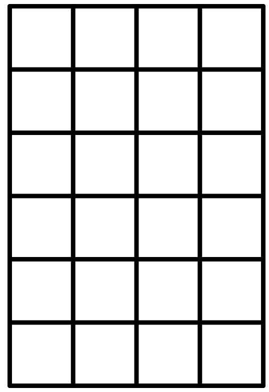
$$N = \sum_{k=1}^{1000} k \left(\lceil \log_{\sqrt{2}} k \rceil - \lfloor \log_{\sqrt{2}} k \rfloor \right).$$

Find the remainder when N is divided by 1000. (Here $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, and $\lceil x \rceil$ denotes the least integer that is greater than or equal to x.)

8. The polynomial $P(x)$ is cubic. What is the largest value of k for which the polynomials $Q_1(x) = x^2 + (k-29)x - k$ and $Q_2(x) = 2x^2 + (2k-43)x + k$ are both factors of $P(x)$?

9. In right triangle ABC with right angle $C, CA = 30$ and $CB = 16$. Its legs $\overline{CA}$ and $\overline{CB}$ are extended beyond A and B. Points O_1 and O_2 lie in the exterior of the triangle and are the centers of two circles with equal radii. The circle with center O_1 is tangent to the hypotenuse and to the extension of leg CA, the circle with center O_2 is tangent to the hypotenuse and to the extension of leg CB, and the circles are externally tangent to each other. The length of the radius of either circle can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

10. In the 6×4 grid shown, 12 of the 24 squares are to be shaded so that there are two shaded squares in each row and three shaded squares in each column. Let N be the number of shadings with this property. Find the remainder when N is divided by $1000.$



11. For each positive integer p, let $b(p)$ denote the unique positive integer k such that $|k - \sqrt{p}| < \frac{1}{2}$. For example, $b(6) = 2$ and $b(23) = 5$. If $S = \sum_{p=1}^{2007} b(p)$, find the remainder when S is divided by $1000.$

12. In isosceles triangle ABC, A is located at the origin and B is located at $(20, 0)$. Point C is in the first quadrant with $AC = BC$ and $\angle BAC = 75^\circ$. If $\triangle ABC$ is rotated counterclockwise about point A until the image of C lies on the positive y-axis, the area of the region common to the original triangle and the rotated triangle is in the form $p\sqrt{2} + q\sqrt{3} + r\sqrt{6} + s$, where p, q, r, s are integers. Find $\frac{p-q+r-s}{2}$.

13. A square pyramid with base $ABCD$ and vertex E has eight edges of length $4.$ A plane passes through the midpoints of $\overline{AE}, \overline{BC}$, and $\overline{CD}$. The plane's intersection with the pyramid has an area that can be expressed as $\sqrt{p}$. Find p.

14. Let a sequence be defined as follows: $a_1 = 3, a_2 = 3$, and for $n \geq 2, a_{n+1}a_{n-1} = a_n^2 + 2007$. Find the largest integer less than or equal to $\frac{a_{2007}^2 + a_{2006}^2}{a_{2007}a_{2006}}$.

15. Let ABC be an equilateral triangle, and let D and F be points on sides BC and AB, respectively, with $FA = 5$ and $CD = 2$. Point E lies on side CA such that $\angle DEF = 60^\circ$. The area of triangle DEF is $14\sqrt{3}$. The two possible values of the length of side AB are $p \pm q\sqrt{r}$, where p and q are rational, and r is an integer not divisible by the square of a prime. Find r.

Solutions: https://live.poshenloh.com/past-contests/aime/2007I/solutions