2006 AIME II

Time limit: 180 minutes

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1. In convex hexagon $ABCDEF$, all six sides are congruent, $\angle A$ and $\angle D$ are right angles, and $\angle B, \angle C, \angle E,$ and $\angle F$ are congruent. The area of the hexagonal region is $2116(\sqrt{2}+1)$. Find AB.

2. The lengths of the sides of a triangle with positive area are $\log_{10} 12, \log_{10} 75,$ and $\log_{10} n,$ where n is a positive integer. Find the number of possible values for n.

3. Let P be the product of the first 100 positive odd integers. Find the largest integer k such that P is divisible by 3^k.

4. Let $(a_1, a_2, a_3, \ldots, a_{12})$ be a permutation of $(1, 2, 3, \ldots, 12)$ for which

$$a_1 > a_2 > a_3 > a_4 > a_5 > a_6 \quad \text{and} \quad a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}.$$

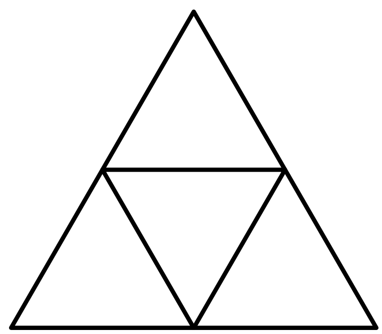
An example of such a permutation is $(6, 5, 4, 3, 2, 1, 7, 8, 9, 10, 11, 12)$. Find the number of such permutations.

5. When rolling a certain unfair six-sided die with faces numbered $1, 2, 3, 4, 5,$ and 6, the probability of obtaining face F is greater than $\frac{1}{6}$, the probability of obtaining the face opposite face F is less than $\frac{1}{6}$, the probability of obtaining each of the other faces is $\frac{1}{6}$, and the sum of the numbers on each pair of opposite faces is 7. When two such dice are rolled, the probability of obtaining a sum of 7 is $\frac{47}{288}$. Given that the probability of obtaining face F is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

6. Square $ABCD$ has sides of length 1. Points E and F are on $\overline{BC}$ and $\overline{CD}$, respectively, so that $\triangle AEF$ is equilateral. A square with vertex B has sides that are parallel to those of $ABCD$ and a vertex on $\overline{AE}$. The length of a side of this smaller square is $\frac{a-\sqrt{b}}{c}$, where $a, b,$ and c are positive integers and b is not divisible by the square of any prime. Find $a + b + c$.

7. Find the number of ordered pairs of positive integers (a, b) such that $a + b = 1000$ and neither a nor b has a zero digit.

8. There is an unlimited supply of congruent equilateral triangles made of colored paper. Each triangle is a solid color with the same color on both sides of the paper. A large equilateral triangle is constructed from four of these paper triangles as shown. Two large triangles are considered distinguishable if it is not possible to place one on the other, using translations, rotations, and/or reflections, so that their corresponding small triangles are of the same color. Given that there are six different colors of triangles from which to choose, how many distinguishable large equilateral triangles can be constructed?



9. Circles $\mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{C}_3$ have their centers at $(0,0)$, $(12,0)$, and $(24,0)$, and have radii 1, 2, and 4, respectively. Line t_1 is a common internal tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$ and has a positive slope, and line t_2 is a common internal tangent to $\mathcal{C}_2$ and $\mathcal{C}_3$ and has a negative slope. Given that lines t_1 and t_2 intersect at (x,y), and that $x = p - q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime, find $p + q + r$.

10. Seven teams play a soccer tournament in which each team plays every other team exactly once. No ties occur, each team has a 50% chance of winning each game it plays, and the outcomes of the games are independent. In each game, the winner is awarded 1 point and the loser gets 0 points. The total points are accumulated to decide the ranks of the teams. In the first game of the tournament, team A beats team B. The probability that team A finishes with more points than team B is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

11. A sequence is defined as follows: $a_1 = a_2 = a_3 = 1$, and, for all positive integers n, $a_{n+3} = a_{n+2} + a_{n+1} + a_n$. Given that $a_{28} = 6090307$, $a_{29} = 11201821$, and $a_{30} = 20603361$, find the remainder when $\sum_{k=1}^{28} a_k$ is divided by 1000.

12. Equilateral $\triangle ABC$ is inscribed in a circle of radius 2. Extend $\overline{AB}$ through B to point D so that $AD = 13$, and extend $\overline{AC}$ through C to point E so that $AE = 11$. Through D, draw a line ℓ_1 parallel to $\overline{AE}$, and through E, draw a line ℓ_2 parallel to $\overline{AD}$. Let F be the intersection of ℓ_1 and ℓ_2. Let G be the point on the circle that is collinear with A and F and distinct from A. Given that the area of $\triangle CBG$ can be expressed in the form $\frac{p\sqrt{q}}{r}$, where $p, q,$ and r are positive integers, p and r are relatively prime, and q is not divisible by the square of any prime, find $p + q + r$.

13. How many integers N less than 1000 can be written as the sum of j consecutive positive odd integers for exactly 5 values of $j \geq 1$?

14. Let S_n be the sum of the reciprocals of the nonzero digits of the integers from 1 to 10^n, inclusive. Find the smallest positive integer n for which S_n is an integer.

15. Given that $x, y,$ and z are real numbers that satisfy

$$x = \sqrt{y^2 - \frac{1}{16}} + \sqrt{z^2 - \frac{1}{16}},$$

$$y = \sqrt{z^2 - \frac{1}{25}} + \sqrt{x^2 - \frac{1}{25}},$$

$$z = \sqrt{x^2 - \frac{1}{36}} + \sqrt{y^2 - \frac{1}{36}},$$

and that $x + y + z = \frac{m}{\sqrt{n}}$, where m and n are positive integers, and n is not divisible by the square of any prime, find $m + n$.

Solutions: https://live.poshenloh.com/past-contests/aime/2006II/solutions