2006 AIME I

Time limit: 180 minutes

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https://live.poshenloh.com/past-contests/aime/2006I

1. In quadrilateral $ABCD$, $\angle B$ is a right angle, diagonal $\overline{AC}$ is perpendicular to $\overline{CD}$, $AB = 18, BC = 21,$ and $CD = 14.$ Find the perimeter of $ABCD.$

2. Let set $\mathcal{A}$ be a 90-element subset of $\{1, 2, 3, \ldots, 100\}$, and let S be the sum of the elements of $\mathcal{A}.$ Find the number of possible values of $S.$

3. Find the least positive integer such that when its leftmost digit is deleted, the resulting integer is $\frac{1}{29}$ of the original integer.

4. Let N be the number of consecutive 0's at the right end of the decimal representation of the product $1!\,2!\,3!\,4!\cdots 99!\,100!$. Find the remainder when N is divided by 1000.

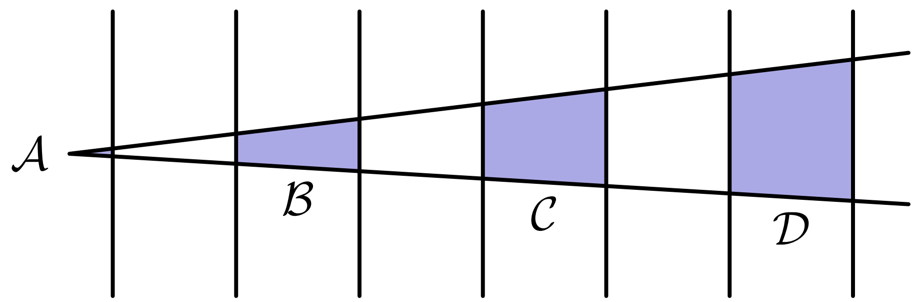
5. The number

$$\sqrt{104\sqrt{6}+468\sqrt{10}+144\sqrt{15}+2006}$$

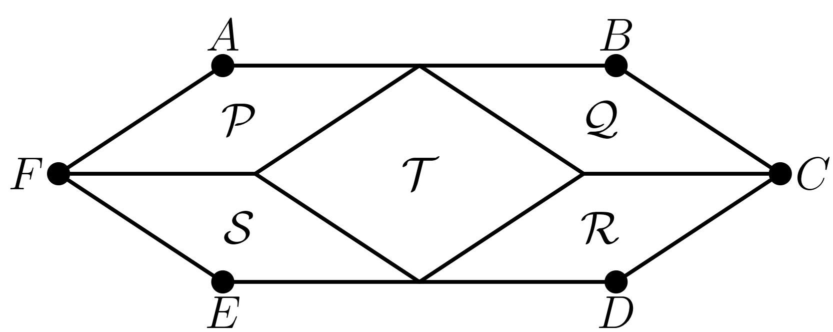
can be written as $a\sqrt{2}+b\sqrt{3}+c\sqrt{5}$, where $a, b,$ and c are positive integers. Find $a \cdot b \cdot c$.

6. Let $\mathcal{S}$ be the set of real numbers that can be represented as repeating decimals of the form $0.\overline{abc}$ where a, b, c are distinct digits. Find the sum of the elements of $\mathcal{S}$.

7. An angle is drawn on a set of equally spaced parallel lines as shown. The ratio of the area of shaded region $\mathcal{C}$ to the area of shaded region $\mathcal{B}$ is $\frac{11}{5}$. Find the ratio of the area of shaded region $\mathcal{D}$ to the area of shaded region $\mathcal{A}$.

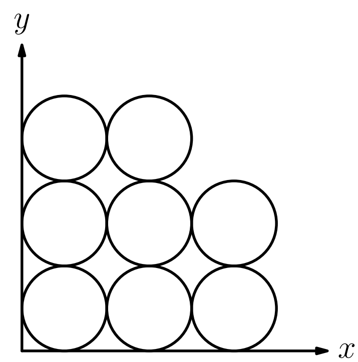


8. Hexagon $ABCDEF$ is divided into five rhombuses, $\mathcal{P}, \mathcal{Q}, \mathcal{R}, \mathcal{S}$, and $\mathcal{T}$, as shown. Rhombuses $\mathcal{P}, \mathcal{Q}, \mathcal{R}$, and $\mathcal{S}$ are congruent, and each has area $\sqrt{2006}$. Let K be the area of rhombus $\mathcal{T}$. Given that K is a positive integer, find the number of possible values for K.



9. The sequence $a_1, a_2, \ldots$ is geometric with $a_1 = a$ and common ratio r, where a and r are positive integers. Given that $\log_8 a_1 + \log_8 a_2 + \cdots + \log_8 a_{12} = 2006$, find the number of possible ordered pairs (a, r).

10. Eight circles of diameter 1 are packed in the first quadrant of the coordinate plane as shown. Let region $\mathcal{R}$ be the union of the eight circular regions. Line ℓ, with slope 3, divides $\mathcal{R}$ into two regions of equal area. Line ℓ's equation can be expressed in the form $ax = by + c$, where a, b, and c are positive integers whose greatest common divisor is 1. Find $a^2 + b^2 + c^2$.



11. A collection of 8 cubes consists of one cube with edge-length k for each integer k, $1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:

• Any cube may be the bottom cube in the tower.

• The cube immediately on top of a cube with edge-length k must have edge-length at most $k+2$.

Let T be the number of different towers that can be constructed. What is the remainder when T is divided by 1000?

12. Find the sum of the values of x such that $\cos^3 3x + \cos^3 5x = 8\cos^3 4x \cos^3 x$, where x is measured in degrees and $100 < x < 200$.

13. For each even positive integer x, let $g(x)$ denote the greatest power of 2 that divides x. For example, $g(20) = 4$ and $g(16) = 16$. For each positive integer n, let $S_n = \sum_{k=1}^{2^{n-1}} g(2k)$. Find the greatest integer n less than 1000 such that S_n is a perfect square.

14. A tripod has three legs each of length 5 feet. When the tripod is set up, the angle between any pair of legs is equal to the angle between any other pair, and the top of the tripod is 4 feet from the ground. In setting up the tripod, the lower 1 foot of one leg breaks off. Let h be the height in feet of the top of the tripod from the ground when the broken tripod is set up. Then h can be written in the form $\frac{m}{\sqrt{n}}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $\lfloor m + \sqrt{n} \rfloor$. (The notation $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x.)

15. Given that a sequence satisfies $x_0 = 0$ and $|x_k| = |x_{k-1} + 3|$ for all integers $k \geq 1$, find the minimum possible value of $|x_1 + x_2 + \cdots + x_{2006}|$.

Solutions: https://live.poshenloh.com/past-contests/aime/2006I/solutions