2005 AIME I

Time limit: 180 minutes

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1. Six congruent circles form a ring with each circle externally tangent to the two circles adjacent to it. All six circles are internally tangent to a circle $\mathcal{C}$ with radius $30.$ Let K be the area of the region inside $\mathcal{C}$ and outside all of the six circles in the ring. Find $\lfloor K \rfloor.$ (The notation $\lfloor K \rfloor$ denotes the greatest integer that is less than or equal to $K.$)

2. For each positive integer $k,$ let S_k denote the increasing arithmetic sequence of integers whose first term is 1 and whose common difference is $k.$ For example, S_3 is the sequence $1, 4, 7, \ldots.$ For how many values of k does S_k contain the term $2005?$

3. How many positive integers have exactly three proper divisors, each of which is less than $50?$ (A *proper divisor* of a positive integer n is a positive integer divisor of n other than n itself.)

4. The director of a marching band wishes to place the members into a formation that includes all of them and has no unfilled positions. If they are arranged in a square formation, there are 5 members left over. The director finds that if they are arranged in a rectangular formation with 7 more rows than columns, the desired result can be obtained. Find the maximum number of members this band can have.

5. Robert has 4 indistinguishable gold coins and 4 indistinguishable silver coins. Each coin has an engraving of a face on one side, but not on the other. He wants to stack the eight coins on a table into a single stack so that no two adjacent coins are face to face. Find the number of possible distinguishable arrangements of the 8 coins.

6. Let P be the product of the nonreal roots of $x^4 - 4x^3 + 6x^2 - 4x = 2005.$ Find $\lfloor P \rfloor.$ (The notation $\lfloor P \rfloor$ denotes the greatest integer that is less than or equal to $P.$)

7. In quadrilateral $ABCD, BC = 8, CD = 12, AD = 10,$ and $m\angle A = m\angle B = 60^\circ.$ Given that $AB = p + \sqrt{q},$ where p and q are positive integers, find $p + q.$

8. The equation

$$2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$$

has three real roots. Given that their sum is $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m + n$.

9. Twenty-seven unit cubes are each painted orange on a set of four faces so that the two unpainted faces share an edge. The 27 cubes are then randomly arranged to form a $3 \times 3 \times 3$ cube. Given that the probability that the entire surface of the larger cube is orange is $\frac{p^a}{q^b r^c}$, where $p, q,$ and r are distinct primes and $a, b,$ and c are positive integers, find $a + b + c + p + q + r$.

10. Triangle ABC lies in the Cartesian plane and has area 70. The coordinates of B and C are $(12, 19)$ and $(23, 20)$, respectively, and the coordinates of A are (p, q). The line containing the median to side $\overline{BC}$ has slope -5. Find the largest possible value of $p + q$.

11. A semicircle with diameter d is contained in a square whose sides have length 8. Given that the maximum value of d is $m - \sqrt{n}$, where m and n are integers, find $m + n$.

12. For positive integers n, let $\tau(n)$ denote the number of positive integer divisors of n, including 1 and n. For example, $\tau(1) = 1$ and $\tau(6) = 4$. Define $S(n)$ by

$$S(n) = \tau(1) + \tau(2) + \cdots + \tau(n).$$

Let a denote the number of positive integers $n \leq 2005$ with $S(n)$ odd, and let b denote the number of positive integers $n \leq 2005$ with $S(n)$ even. Find $|a - b|$.

13. A particle moves in the Cartesian plane from one lattice point to another according to the following rules:

• From any lattice point (a, b), the particle may move only to $(a + 1, b)$, $(a, b + 1)$, or $(a + 1, b + 1)$.

• There are no right angle turns in the particle's path. That is, the sequence of points visited contains neither a subsequence of the form $(a, b), (a + 1, b), (a + 1, b + 1)$ nor a subsequence of the form $(a, b), (a, b + 1), (a + 1, b + 1)$.

How many different paths can the particle take from $(0, 0)$ to $(5, 5)$?

14. Consider the points $A(0, 12), B(10, 9), C(8, 0)$, and $D(-4, 7)$. There is a unique square $\mathcal{S}$ such that each of the four points is on a different side of $\mathcal{S}$. Let K be the area of $\mathcal{S}$. Find the remainder when $10K$ is divided by 1000.

15. In $\triangle ABC, AB = 20.$ The incircle of the triangle divides the median containing C into three segments of equal length. Given that the area of $\triangle ABC$ is $m\sqrt{n},$ where m and n are integers and n is not divisible by the square of any prime, find $m + n.$

Solutions: https://live.poshenloh.com/past-contests/aime/2005I/solutions