

# 1997 AIME

Time limit: 180 minutes

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1. How many of the integers between 1 and 1000, inclusive, can be expressed as the difference of the squares of two nonnegative integers?
2. The nine horizontal and nine vertical lines on an  $8 \times 8$  checkerboard form  $r$  rectangles, of which  $s$  are squares. The number  $s/r$  can be written in the form  $m/n$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
3. Sarah intended to multiply a two-digit number and a three-digit number, but she left out the multiplication sign and simply placed the two-digit number to the left of the three-digit number, thereby forming a five-digit number. This number is exactly nine times the product Sarah should have obtained. What is the sum of the two-digit number and the three-digit number?

4. Circles of radii  $5$ ,  $5$ ,  $8$ , and  $\frac{m}{n}$  are mutually externally tangent, where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
5. The number  $r$  can be expressed as a four-place decimal  $0.abcd$ , where  $a$ ,  $b$ ,  $c$ , and  $d$  represent digits, any of which could be zero. It is desired to approximate  $r$  by a fraction whose numerator is  $1$  or  $2$  and whose denominator is an integer. The closest such fraction to  $r$  is  $\frac{2}{7}$ . What is the number of possible values for  $r$ ?
6. Point  $B$  is in the exterior of the regular  $n$ -sided polygon  $A_1A_2 \cdots A_n$ , and  $A_1A_2B$  is an equilateral triangle. What is the largest value of  $n$  for which  $A_n$ ,  $A_1$ , and  $B$  are consecutive vertices of a regular polygon?
7. A car travels due east at  $\frac{2}{3}$  mile per minute on a long, straight road. At the same time, a circular storm, whose radius is  $51$  miles, moves southeast at  $\frac{1}{2}\sqrt{2}$  mile per minute. At time  $t = 0$ , the center of the storm is  $110$  miles due north of the car. At time  $t = t_1$  minutes, the car enters the storm circle, and at time  $t = t_2$  minutes, the car leaves the storm circle. Find  $\frac{1}{2}(t_1 + t_2)$ .

8. How many different  $4 \times 4$  arrays whose entries are all 1's and  $-1$ 's have the property that the sum of the entries in each row is 0 and the sum of the entries in each column is 0?

9. Given a nonnegative real number  $x$ , let  $\langle x \rangle$  denote the fractional part of  $x$ ; that is,  $\langle x \rangle = x - \lfloor x \rfloor$ , where  $\lfloor x \rfloor$  denotes the greatest integer less than or equal to  $x$ . Suppose that  $a$  is positive,  $\langle a^{-1} \rangle = \langle a^2 \rangle$ , and  $2 < a^2 < 3$ . Find the value of  $a^{12} - 144a^{-1}$ .

10. Every card in a deck has a picture of one shape — circle, square, or triangle, which is painted in one of the three colors — red, blue, or green. Furthermore, each color is applied in one of three shades — light, medium, or dark. The deck has 27 cards, with every shape-color-shade combination represented. A set of three cards from the deck is called *complementary* if all of the following statements are true:

- Either each of the three cards has a different shape or all three of the cards have the same shape.
- Either each of the three cards has a different color or all three of the cards have the same color.
- Either each of the three cards has a different shade or all three of the cards have the same shade.

How many different complementary three-card sets are there?

11. Let

$$x = \frac{\sum_{n=1}^{44} \cos n^\circ}{\sum_{n=1}^{44} \sin n^\circ}.$$

What is the greatest integer that does not exceed  $100x$ ?

12. The function  $f$  defined by  $f(x) = \frac{ax+b}{cx+d}$ , where  $a, b, c,$  and  $d$  are nonzero real numbers, has the properties  $f(19) = 19, f(97) = 97,$  and  $f(f(x)) = x$  for all values except  $\frac{-d}{c}$ . Find the unique number that is not in the range of  $f$ .

13. Let  $S$  be the set of points in the Cartesian plane that satisfy

$$\left| \left| |x| - 2 \right| - 1 \right| + \left| \left| |y| - 2 \right| - 1 \right| = 1.$$

If a model of  $S$  were built from wire of negligible thickness, then the total length of wire required would be  $a\sqrt{b}$ , where  $a$  and  $b$  are positive integers and  $b$  is not divisible by the square of any prime number. Find  $a + b$ .

14. Let  $v$  and  $w$  be distinct, randomly chosen roots of the equation  $z^{1997} - 1 = 0$ . Let  $\frac{m}{n}$  be the probability that  $\sqrt{2 + \sqrt{3}} \leq |v + w|$ , where  $m$  and  $n$  are relatively prime positive integers. Find  $m + n$ .
15. The sides of rectangle  $ABCD$  have lengths 10 and 11. An equilateral triangle is drawn so that no point of the triangle lies outside  $ABCD$ . The maximum possible area of such a triangle can be written in the form  $p\sqrt{q} - r$ , where  $p$ ,  $q$ , and  $r$  are positive integers, and  $q$  is not divisible by the square of any prime number. Find  $p + q + r$ .

Solutions: <https://live.poshenloh.com/past-contests/aime/1997/solutions>

