2017 AMC 8 Solutions

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1. Which of the following values is largest?

A $2+0+1+7$

B $2\times 0+1+7$

C $2+0\times 1+7$

D $2+0+1\times 7$

E $2\times 0\times 1\times 7$

Solution(s):

Option (A) evaluates to 10.

Option (B) evaluates to $0+1+7=8$.

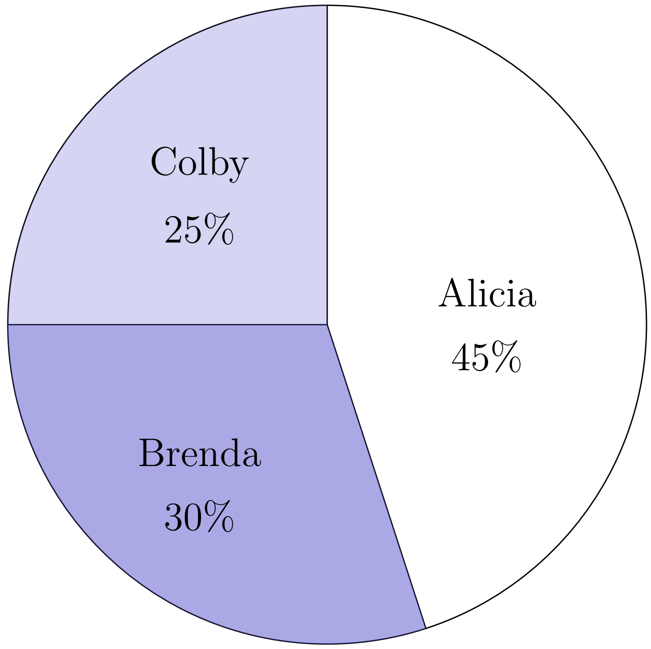
Option (C) evalutes to $2+0+7=9$.

Option (D) evaluates to $2+0+7=9$.

Option (E) evalutes to 0.

Thus, **A** is the correct answer.

2. Alicia, Brenda, and Colby were the candidates in a recent election for student president. The pie chart below shows how the votes were distributed among the three candidates. If Brenda received 36 votes, then how many votes were cast all together?



A 70

B 84

C 100

D 106

E 120

Solution(s):

If 36 votes is 30% of the total votes, then 10% of the total votes is 12 votes. The number of total votes would then be $10 \cdot 12 = 120$.

Thus, **E** is the correct answer.

3. What is the value of the expression $\sqrt{16\sqrt{8\sqrt{4}}}$?

A 4

B $4\sqrt{2}$

C 8

D $8\sqrt{2}$

E 16

Solution(s):

This expression can be reduced as follows:

$$\begin{aligned}\sqrt{16\sqrt{8\sqrt{4}}} &= \sqrt{16\sqrt{16}} \\ &= \sqrt{64} \\ &= 8\end{aligned}$$

Thus, **C** is the correct answer.

4. When 0.000315 is multiplied by $7,928,564$ the product is closest to which of the following?

A 210

B 240

C 2100

D 2400

E 24000

Solution(s):

We can approximate the product as

$$\begin{aligned}(3\cdot 10^{-4})(8\cdot 10^{6}) &= 24\cdot 10^{2}\\ &= 2400.\end{aligned}$$

Thus, **D** is the correct answer.

5. What is the value of the expression

$$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8}?$$

A 1020

B 1120

C 1220

D 2240

E 3360

Solution(s):

Looking at the denominator independently, $1 + 2 + 3 + \cdots + 8 = 36$, so the desired answer is

$$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{36} = 4 \cdot 5 \cdot 7 \cdot 8 = 1120.$$

Thus, **B** is the correct answer.

6. If the degree measures of the angles of a triangle are in the ratio $3 : 3 : 4$, what is the degree measure of the largest angle of the triangle?

A 18

B 36

C 60

D 72

E 90

Solution(s):

We can let the three angles be equal to $3x, 3x,$ and $4x$. Then we know that their sum equals 180. From this we can set $3x + 3x + 4x = 180$, and solving this, we get $10x = 180$ and $x = 18$.

The largest angle is $4x = 72$.

Thus, **D** is the correct answer.

7. Let Z be a 6-digit positive integer, such as 247247, whose first three digits are the same as its last three digits taken in the same order. Which of the following numbers must also be a factor of Z?

A 11

B 19

C 101

D 111

E 1111

Solution(s):

We can let Z have the form $abcabc$. Then, we get that

$$Z = 1001 \cdot abc = 7 \cdot 11 \cdot 13 \cdot abc.$$

This means that 11 is a factor of Z.

Thus, **A** is the correct answer.

8. Malcolm wants to visit Isabella after school today and knows the street where she lives but doesn't know her house number. She tells him, "My house number has two digits, and exactly three of the following four statements about it are true."

(1) It is prime.

(2) It is even.

(3) It is divisible by 7.

(4) One of its digits is 9.

This information allows Malcolm to determine Isabella's house number. What is its units digit?

A 4

B 6

C 7

D 8

E 9

Solution(s):

If a number is even, it is divisible by 2. This means that the house number is either divisible by 2 or by 7. This rules out the possibility that the house number is prime. This means that the house number is divisible by 2 and by 7, and one of its digits is 9.

The divisibility conditions show that the house number is divisble by 14, so if we look at all the two-digit multiples of 14, the only one with a digit of 9 is 98. Therefore, units digit is 8.

Thus, **D** is the correct answer.

9. All of Marcy's marbles are blue, red, green, or yellow. One third of her marbles are blue, one fourth of them are red, and six of them are green. What is the smallest number of yellow marbles that Macy could have?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution(s):

If the number of marbles is divisble by both 3 and 4, then the number must be divisible by 12. If we test 12, we get that there are 4 blue marbles and 3 red marbles. This leaves a maximum of 5 green marbles, which is not possible.

If there are 24 marbles, then there are 8 blue marbles and 6 red marbles. To find the number of yellow marbles, we get $24 - 8 - 6 - 6 = 4$.

Thus, **D** is the correct answer.

10. A box contains five cards, numbered 1, 2, 3, 4, and 5. Three cards are selected randomly without replacement from the box. What is the probability that 4 is the largest value selected?

A $\frac{1}{10}$

B $\frac{1}{5}$

C $\frac{3}{10}$

D $\frac{2}{5}$

E $\frac{1}{2}$

Solution(s):

The number of ways to choose 3 cards from 5 is $\binom{5}{3} = 10$. If 4 is the largest value selected, then the other two cards have to be chosen from $1, 2, 3$. There are $\binom{3}{2} = 3$ ways to do this. The probability is then $\frac{3}{10}$.

Thus, **C** is the correct answer.

11. A square-shaped floor is covered with congruent square tiles. If the total number of tiles that lie on the two diagonals is 37, how many tiles cover the floor?

- A 148
- B 324
- C 361
- D 1296
- E 1369

Solution(s):

37 tiles on both diagonals imply that there are 19 tiles on each diagonal, since one tile overlaps in the middle. The total number of tiles would then be $19^2 = 361$ since the number of tiles in each row is equal to the number of tiles in one diagonal.

Thus, **C** is the correct answer.

12. The smallest positive integer greater than 1 that leaves a remainder of 1 when divided by 4, 5, and 6 lies between which of the following pairs of numbers?

A $2 \text{ and } 19$

B $20 \text{ and } 39$

C $40 \text{ and } 59$

D $60 \text{ and } 79$

E $80 \text{ and } 124$

Solution(s):

If a number leaves a remainder of 1 when divided by $4, 5, \text{ and } 6,$ then it is one more than the least common multiple of these numbers. The least common multiple is $60,$ so the smallest such positive integer is $60 + 1 = 61.$

Thus, **D** is the correct answer.

13. Peter, Emma, and Kyler played chess with each other. Peter won 4 games and lost 2 games. Emma won 3 games and lost 3 games. If Kyler lost 3 games, how many games did he win?

A 0

B 1

C 2

D 3

E 4

Solution(s):

Since there are no ties, the number of wins has to equal the number of losses. The number of losses is $2+3+3=8$, so the number of games Kyler won is $8-4-3=1$.

Thus, **B** is the correct answer.

14. Chloe and Zoe are both students in Ms. Demeanor's math class. Last night they each solved half of the problems in their homework assignment alone and then solved the other half together. Chloe had correct answers to only 80% of the problems she solved alone, but overall 88% of her answers were correct. Zoe had correct answers to 90% of the problems she solved alone. What was Zoe's overall percentage of correct answers?

A 89

B 92

C 93

D 96

E 98

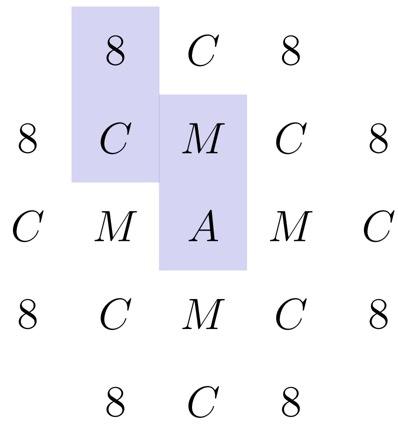
Solution(s):

Since the answer should be same regardless of the number of problems, we can assume that there were 100 problems on the assignment. 80% of 50 is 40, so Chloe answered 40 questions correctly alone. 88% of 100 is 88, so Chloe answered 88 questions correctly in total. This means that Chloe answered $88 - 40 = 48$ together with Zoe.

90% of 50 is 45, so Zoe answered 45 questions correct by herself. We know that she answered 48 questions correct with Chloe, so she answered $45 + 48 = 93$ correctly in total. This means that her overall percentage is 93%.

Thus, **C** is the correct answer.

15. In the arrangement of letters and numerals below, by how many different paths can one spell AMC8? Beginning at the A in the middle, a path only allows moves from one letter to an adjacent (above, below, left, or right, but not diagonal) letter. One example of such a path is traced in the picture.



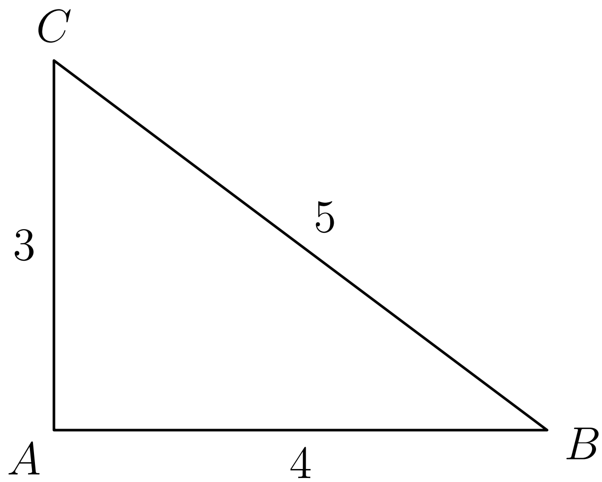
- **A** 8
- **B** 9
- **C** 12
- **D** 24
- **E** 36

Solution(s):

Starting from A, there are 4 ways to reach an M. From each M, there are 3 ways to reach a C. From each C, there are 2 ways to reach an 8. Multiplying all these possibilities, we get $4 \cdot 3 \cdot 2 = 24$.

Thus, **D** is the correct answer.

16. In the figure below, choose point D on $\overline{BC}$ so that $\triangle ACD$ and $\triangle ABD$ have equal perimeters. What is the area of $\triangle ABD$?



A $\frac{3}{4}$

B $\frac{3}{2}$

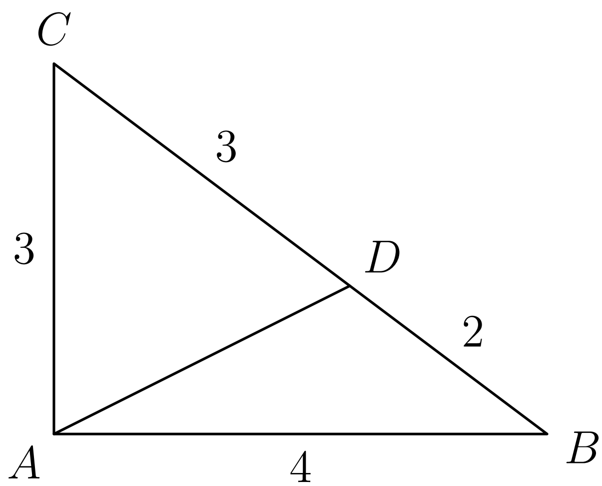
C 2

D $\frac{12}{5}$

E $\frac{5}{2}$

Solution(s):

The only way to split $\overline{BC}$ into two parts such that the two triangles have the same perimeter is if $\overline{CD} = 3$ and $\overline{BD} = 2$.



$\triangle ACD$ and $\triangle ABD$ have the same altitudes, so their areas are proportional to their bases. This means that the area of $\triangle ABD$ is $\frac{2}{5}$ the area of $\triangle ABC$, which is

$$\frac{2}{5} \cdot 3 \cdot \frac{4}{2} = \frac{12}{5}.$$

Thus, **D** is the correct answer.

17. Starting with some gold coins and some empty treasure chests, I tried to put 9 gold coins in each treasure chest, but that left 2 treasure chests empty. So instead I put 6 gold coins in each treasure chest, but then I had 3 gold coins left over. How many gold coins did I have?

A 9

B 27

C 45

D 63

E 81

Solution(s):

Let n be the number of treasure chests and g be the number of gold coins. Then

$$9(n-2)=g$$

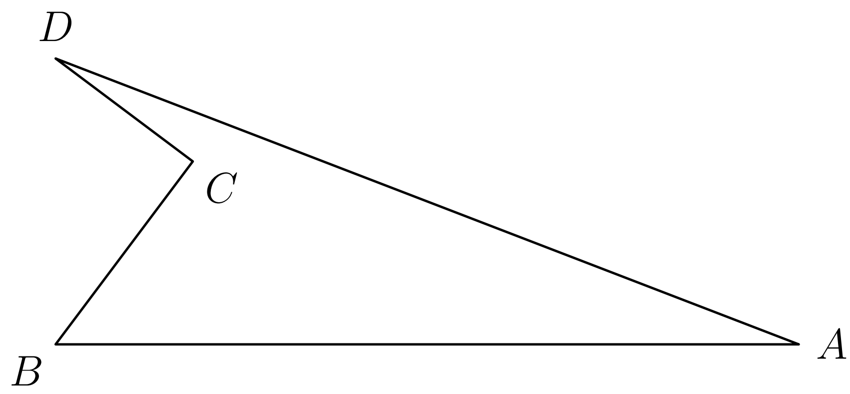
and

$$6n+3=g.$$

Solving this system yields $n=7$, so the number of gold coins is $6\cdot 7+3=45$.

Thus, **C** is the correct answer.

18. In the non-convex quadrilateral $ABCD$ shown below, $\angle BCD$ is a right angle, $AB = 12$, $BC = 4$, $CD = 3$, and $AD = 13$. What is the area of quadrilateral $ABCD$?



A 12

B 24

C 26

D 30

E 36

Solution(s):

Since $\angle BCD$ is a right angle, we can apply the Pythagorean theorem to $\triangle BCD$ to get that $\overline{BD} = 5$. We also get that $\angle DBA$ is right since the sides of $\triangle BDA$ form a Pythagorean triple.

Then the area of $ABCD$ is equal to

$$\begin{aligned} \text{area}(\triangle BDA) - \ \text{area}(\triangle BCD) \\ &= \frac{1}{2} \cdot 12 \cdot 5 - \frac{1}{2} \cdot 4 \cdot 3 \\ &= 30 - 6 \\ &= 24. \end{aligned}$$

Thus, **B** is the correct answer.

19. For any positive integer M, the notation $M!$ denotes the product of the integers 1 through M. What is the largest integer n for which 5^n is a factor of the sum:

$$98! + 99! + 100!$$

A 23

B 24

C 25

D 26

E 27

Solution(s):

The expression can be factored as follows:

$$\begin{aligned}&98! + 99! + 100!\\ &= 98! + 99 \cdot 98! + 100 \cdot 99 \cdot 98!\\ &= 98!(1 + 99 + 100 \cdot 99)\\ &= 98!(100 + 100 \cdot 99)\\ &= 98! \cdot 100 \cdot (1 + 99)\\ &= 98! \cdot 100^2\end{aligned}$$

Each 100 possesses two factors of 5. The number of factors of 5 in $98!$ is $\lfloor 98/5 \rfloor + \lfloor 98/25 \rfloor = 19 + 3 = 22$. This counts the number of numbers divisible by 5 and the number of numbers divisible by 25 to get both factors of 5. The total number of factors of 5 is $2 + 2 + 22 = 26$.

Thus, **D** is the correct answer.

20. An integer between 1000 and 9999, inclusive, is chosen at random. What is the probability that it is an odd integer whose digits are all distinct?

A $\frac{14}{75}$

B $\frac{56}{225}$

C $\frac{107}{400}$

D $\frac{7}{25}$

E $\frac{9}{25}$

Solution(s):

Since the number is odd, the last digit is odd, giving 5 possibilities. The thousands digit cannot be zero or the the number we already got, so that gives 8 possibilities. Similarly, the hundreds digit has 8 possibilities, and the tens digit has 7 possibilities. This gives a total of $5 \cdot 8 \cdot 8 \cdot 7 = 2240$, making the probability $\frac{2240}{9000} = \frac{56}{225}$.

Thus, **B** is the correct answer.

21. Suppose a, b, and c are nonzero real numbers, and $a+b+c=0$. What are the possible value(s) for

$$\frac{a}{|a|}+\frac{b}{|b|}+\frac{c}{|c|}+\frac{abc}{|abc|}?$$

A 0

B $1 \text{ and } -1$

C $2 \text{ and } -2$

D $0, 2, \text{ and } -2$

E $0, 1, \text{ and } -1$

Solution(s):

Since the sum is 0, all the numbers cannot be positive or negative. This gives two cases: two are positive and one is negative or vice versa. Also note that if $x<0$, $\frac{x}{|x|}=-1$, and if $x>0$, $\frac{x}{|x|}=1$.

Case I: two are positive and one is negative

WLOG, let $a, b>0$ and $c<0$. Then $abc<0$. This means that $\frac{a}{|a|}=\frac{b}{|b|}=1$ and $\frac{c}{|c|}=\frac{abc}{|abc|}=-1$. Then

$$\frac{a}{|a|}+\frac{b}{|b|}+\frac{c}{|c|}+\frac{abc}{|abc|}=0.$$

Case II: two are negative and one is positive

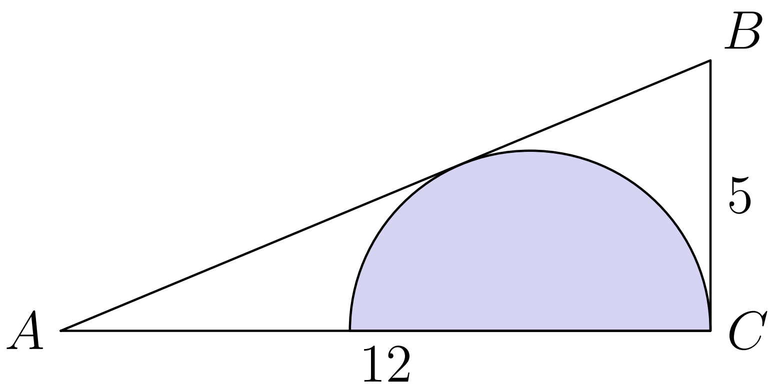
WLOG, let $a, b<0$ and $c>0$. Then $abc>0$. This means that $\frac{a}{|a|}=\frac{b}{|b|}=-1$ and $\frac{c}{|c|}=\frac{abc}{|abc|}=1$. Then

$$\frac{a}{|a|}+\frac{b}{|b|}+\frac{c}{|c|}+\frac{abc}{|abc|}=0.$$

Either way, $\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} + \frac{abc}{|abc|}$ equals 0.

Thus, **A** is the correct answer.

22. In the right triangle ABC, $AC = 12$, $BC = 5$, and angle C is a right angle. A semicircle is inscribed in the triangle as shown. What is the radius of the semicircle?



A $\frac{7}{6}$

B $\frac{13}{5}$

C $\frac{59}{18}$

D $\frac{10}{3}$

E $\frac{60}{13}$

Solution(s):

Let O be the center of the inscribed semicircle and D be the tangent point of the semicircle on $\overline{AB}$. Then $BD = 5$ since $\overline{BD}$ and $\overline{BC}$ are tangents to the semicircle. Then $AD = 8$ and $OD = r$. $\overline{OD}$ is perpendicular to$\overline{AB}$ so $\triangle ADO \sim \triangle BCA$, so $\frac{r}{8} = \frac{5}{12}$. Solving this, we get $r = \frac{10}{3}$.

Thus, **D** is the correct answer.

23. Each day for four days, Linda traveled for one hour at a speed that resulted in her traveling one mile in an integer number of minutes. Each day after the first, her speed decreased so that the number of minutes to travel one mile increased by 5 minutes over the preceding day. Each of the four days, her distance traveled was also an integer number of miles. What was the total number of miles for the four trips?

A 10

B 15

C 25

D 50

E 82

Solution(s):

Linda traveled for 60 minutes every day. Since one mile was traveled in an integer amount of minutes each day, her mph every day must be a factor of 60. The factors of 60 are $1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30$, and 60. The only sequence of four of these numbers that differ by 5 are $5, 10, 15$, and 20. For the four days, she traveled $\frac{60}{5} + \frac{60}{10} + \frac{60}{15} + \frac{60}{20} = 25$ miles.

Thus, **C** is the correct answer.

24. Mrs. Sanders has three grandchildren, who call her regularly. One calls her every three days, one calls her every four days, and one calls her every five days. All three called her on December 31, 2016. On how many days during the next year did she not receive a phone call from any of her grandchildren?

A 78

B 80

C 144

D 146

E 152

Solution(s):

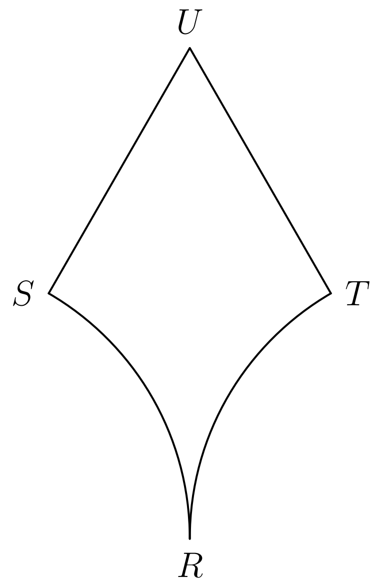
In a 60 day period, the first child calls 20 times, the second child calls 15 times, and the third child calls 12 days. $20+15+12=47$ overcounts, however. The first and second children call on the same day $60/12=5$ times. The first and third children call on the same day $60/15=4$ times. The second and third children call on the same day $60/20=3$ times. Subtracting these from 47 yields $47-5-4-3=35$.

The 60^{th} is added in thrice and subtracted out thrice, so we need to add it back in. This means that for every 60 days, Mrs. Sanders receives a call 36 days, which means that she does not receive a call on 24 days. There are 6 60 day periods, and there are no calls the 361^{st} or 362^{nd} day, which results in $24\cdot 6+2=146$ total days with no phone calls.

Thus, **D** is the correct answer.

25. In the figure shown, $\overline{US}$ and $\overline{UT}$ are line segments each of length 2, and $m\angle TUS = 60^\circ$.

Arcs $\widehat{TR}$ and $\widehat{SR}$ are each one-sixth of a circle with radius 2. What is the area of the region shown?



A $3\sqrt{3} - \pi$

B $4\sqrt{3} - \frac{4\pi}{3}$

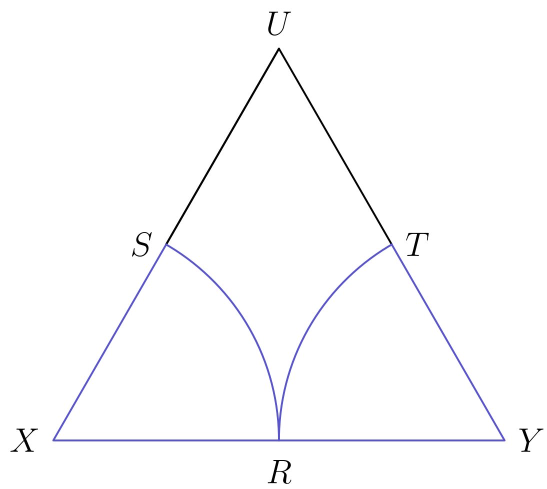
C $2\sqrt{3}$

D $4\sqrt{3} - \frac{2\pi}{3}$

E $4 + \frac{4\pi}{3}$

Solution(s):

We can extend $\overline{SU}$ and $\overline{TU}$ to form the following picture.



The area of this region is the area of an equilaterial triangle with side length of 4 minus the area of two-sixths of a circle with radius 2. The area for an equilateral triangle with side length s is $\frac{s^2\sqrt{3}}{4}$. This means that the total area is

$$\frac{4^2\sqrt{3}}{4} - \frac{1}{3}\pi \cdot 2^2 = 4\sqrt{3} - \frac{4}{3}\pi.$$

Thus, **B** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc8/2017