2015 AMC 8 Solutions

Typeset by: LIVE, by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc8/2015/solutions

1. How many square yards of red carpet are required to cover a rectangular floor that is 12 feet long and 9 feet wide? (There are 3 feet in a yard.)

A 12

B 36

C 108

D 324

E 972

Solution(s):

Since one side is 12 feet, it would be $\frac{12}{3} = 4$ yards.

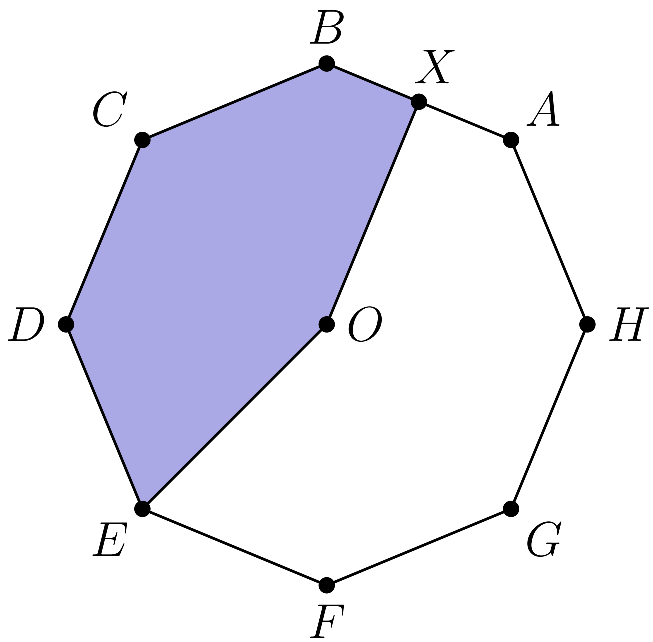
Since another side is 9 feet, it would be $\frac{9}{3} = 3$ yards.

Since the dimensions are $4 \text{ yards} \times 3 \text{ yards}$, the area is equal to

$$4 \cdot 3 = 12.$$

Thus, the correct answer is **A** .

2. Point O is the center of the regular octagon $ABCDEFGH$, and X is the midpoint of the side $\overline{AB}$. What fraction of the area of the octagon is shaded?



A $\frac{11}{32}$

B $\frac{3}{8}$

C $\frac{13}{32}$

D $\frac{7}{16}$

E $\frac{15}{32}$

Solution(s):

First notice that there are 8 equally sized triangles that can be created with O and any two consecutive points. Therefore, they each take up $\frac{1}{8}$ of the total area of the octagon.

The shaded area has three complete triangles and half of the triangle ABO. Therefore, the shaded area is

$$\frac{3.5}{8} = \frac{7}{16}$$

of the total area of the octagon.

Thus, the correct answer is **D**.

3. Jack and Jill are going swimming at a pool that is one mile from their house. They leave home simultaneously. Jill rides her bicycle to the pool at a constant speed of 10 miles per hour. Jack walks to the pool at a constant speed of 4 miles per hour. How many minutes before Jack does Jill arrive?

A 5

B 6

C 8

D 9

E 10

Solution(s):

Jack travels at a rate of 4 miles per 60 minutes. Therefore, it takes him $\frac{60}{4} = 15$ minutes to get to the pool.

Jill travels at a rate of 10 miles per 60 minutes. Therefore it takes her $\frac{60}{10} = 6$ minutes to get to the pool.

Therefore, the difference in their times is $15 - 6 = 9$ minutes.

Thus, the correct answer is **D**.

4. The Centerville High School chess team consists of two boys and three girls. A photographer wants to take a picture of the team to appear in the local newspaper. She decides to have them sit in a row with a boy at each end and the three girls in the middle. How many such arrangements are possible?

A 2

B 4

C 5

D 6

E 12

Solution(s):

Since there are 2 boys and two places to put them, there are $2! = 2$ ways to place them.

Since there are 3 boys and three places to put them, there are $3! = 6$ ways to place them.

Therefore, the total number of places to put them is $6 \cdot 2 = 12$.

Thus, the correct answer is **E**.

5. Billy's basketball team scored the following points over the course of the first 11 games of the season:

$$42, 47, 53, 53, 58, 58,$$

$$58, 61, 64, 65, 73.$$

If his team scores 40 in the 12^{th} game, which of the following statistics will show an increase?

A range

B median

C mean

D mode

E mid-range

Solution(s):

When considering all 12 games, 40 -- from the 12^{th} game -- will be the lowest score. Therefore, compared to the range of just the first 11 games, the range of all 12 games would increase from $73 - 42 = 31$ to $73 - 40 = 33.$

Thus, the correct answer is **A** .

6. In $\triangle ABC$, $AB = BC = 29$, and $AC = 42$. What is the area of $\triangle ABC$?

A 100

B 420

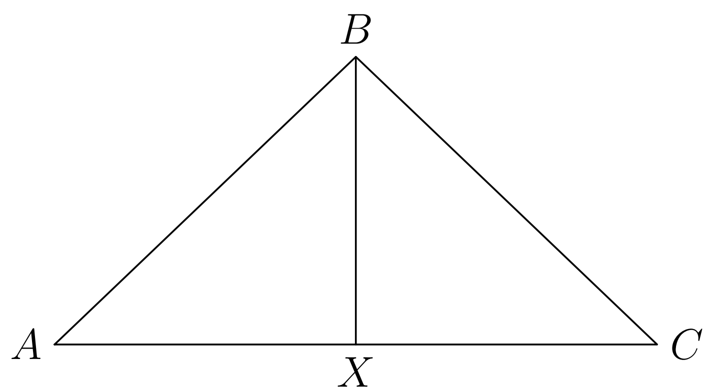
C 500

D 609

E 701

Solution(s):

Let's begin by sketching the triangle:



Define X to be the midpoint of AC. Therefore, $AX = 21$.

Then, since $AB = BC$, we know that AXB is a right triangle. By the Pythagorean Theorem, we know $AX^2 + BX^2 = AB^2$, so we get

$$\begin{aligned} BX &= \sqrt{AB^2 - BX^2} \\ &= \sqrt{29^2 - 21^2} \\ &= \sqrt{400} = 20. \end{aligned}$$

Then, as the area of a triangle is equal to $\frac{1}{2}bh$ formula for area, we can see that the area is $\frac{42 \cdot 20}{2} = 420.$

Thus, the correct answer is **B**.

7. Each of two boxes contains three chips numbered $1, 2, 3$. A chip is drawn randomly from each box and the numbers on the two chips are multiplied. What is the probability that their product is even?

A $\frac{1}{9}$

B $\frac{2}{9}$

C $\frac{4}{9}$

D $\frac{1}{2}$

E $\frac{5}{9}$

Solution(s):

First, let us try to find the probability that the product is odd. We know that the product will be odd if both boxes yield odd numbers. As there is a $\frac{2}{3}$ chance of a box being odd, there is a

$$\left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

chance of the product being odd.

The probability that a number is even is 1 minus the probability that it is odd (as they are complements), so the odds of it being even is

$$1 - \frac{4}{9} = \frac{5}{9}.$$

Thus, the correct answer is **E**.

8. What is the smallest whole number larger than the perimeter of any triangle with a side of length 5 and a side of length 19?

A 24

B 29

C 43

D 48

E 57

Solution(s):

Let the third side be s. By the triangle inequality, we know $s < 5 + 19$. Therefore, the perimeter of the triangle:

$$s + 5 + 19 < 2(5 + 19) = 48.$$

As such, the largest triangle we can make under these constraints is a triangle with sides $5, 19, 23$, and therefore a perimeter of 47, as required. Therefore, the smallest possible number greater than any perimeter is 48.

Thus, the correct answer is **D**.

9. On her first day of work, Janabel sold one widget. On day two, she sold three widgets. On day three, she sold five widgets, and on each succeeding day, she sold two more widgets than she had sold on the previous day. How many widgets in total had Janabel sold after working 20 days?

A 39

B 40

C 210

D 400

E 401

Solution(s):

We want to find

$$1 + 3 + \cdots + 39.$$

This sum can be rewritten as

$$(1 + 39) + (3 + 37) \\ + \cdots + (19 + 21),$$

which we can see has 10 terms. Further notice that each term is equal to 40. Therefore, the sum is

$$10 \cdot 40 = 400.$$

Thus, the correct answer is **D** .

10. How many integers between 1000 and 9999 have four distinct digits?

- A 3024
- B 4536
- C 5040
- D 6480
- E 6561

Solution(s):

First, there are 9 digits to choose for the thousands digit since 0 can't be chosen.

Then, after that, there are 9 ways to choose the hundreds digit, 8 ways to choose the tens digit, and 7 ways to choose the ones digit. Therefore, we get $9 \cdot 9 \cdot 8 \cdot 7 = 4536$ ways to choose such an integer.

Thus, the correct answer is **B** .

11. In the small country of Mathland, all automobile license plates have four symbols. The first must be a vowel (A, E, I, O, or U), the second and third must be two different letters among the 21 non-vowels, and the fourth must be a digit (0 through 9). If the symbols are chosen at random subject to these conditions, what is the probability that the plate will read "AMC8"?

A $\frac{1}{22,050}$

B $\frac{1}{21,000}$

C $\frac{1}{10,500}$

D $\frac{1}{2,100}$

E $\frac{1}{1,050}$

Solution(s):

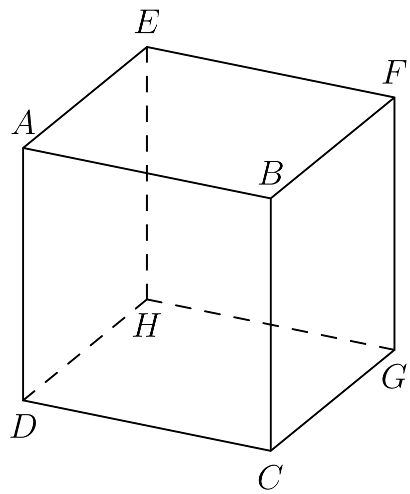
First, we find how many license plates are possible.

For the first letter, we have 5 choices. For the second letter, we have 21 choices. For the third letter, we have $21 - 1 = 20$ choices since we must exlude the letter chosen for the second letter. For the final letter, we have 10 choices. Overall, there are $5 \cdot 21 \cdot 20 \cdot 10 = 21,000$ choices.

The choice AMC8 is one possible choice of these $21,000$ possibilites, so the probability of it being our choice is $\frac{1}{21,000}$.

Thus, the correct answer is **B**.

12. How many pairs of parallel edges, such as $\overline{AB}$ and $\overline{GH}$ or $\overline{EH}$ and $\overline{FG}$, does a cube have?



- A 6
- B 12
- C 18
- D 24
- E 36

Solution(s):

We have 12 edges, and each edge has 3 parallel lines to it. However, this would double count it, so we divide by 2. Therefore, there are $\frac{3 \cdot 12}{2} = 18$ set of parallel edges.

Thus, the correct answer is **C**.

13. How many subsets of two elements can be removed from the set

$$\{1,2,3,4,5,6,7,8,9,10,11\}$$

so that the mean (average) of the remaining numbers is 6?

A 1

B 2

C 3

D 5

E 6

Solution(s):

Currently, the sum is 66. If we remove 2 elements, then there are 9 elements remaining. Since their mean is 6, the sum of the remaining elements is 54. This means the sum of the two elements we take out are $66-54=12$. There are 5 two elements sets that work, namely:

$$\{1,11\}$$
$$\{2,10\}$$
$$\{3,9\}$$
$$\{4,8\}$$
$$\{5,7\}$$

Thus, the correct answer is **D**.

14. Which of the following integers cannot be written as the sum of four consecutive odd integers?

A 16

B 40

C 72

D 100

E 200

Solution(s):

Let $2k+1$ be the lowest of the odd integers in question. Then, the next three consecutive odd integers are $2k+3, 2k+5$, and $2k+7$.

It follows that their sum is $8k+16=8(k+2)$. This means the integer must a multiple of 8. The only answer choice that is not divisible by 8 is 100.

Thus, the correct answer is **D.**

15. At Euler Middle School, 198 students voted on two issues in a school referendum with the following results: 149 voted in favor of the first issue and 119 voted in favor of the second issue. If there were exactly 29 students who voted against both issues, how many students voted in favor of both issues?

A 49

B 70

C 79

D 99

E 149

Solution(s):

Since 29 students voted against both, we know that $198 - 29 = 169$ people voted for at least one.

As we know that 149 students voted for the first issue, and 119 students voted for the second issue, and 169 students that voted for at least one issue, we conclude that the number of students that voted for both is

$$149 + 119 - 169 = 99.$$

Thus, the correct answer is **D**.

16. In a middle-school mentoring program, a number of the sixth graders are paired with a ninth-grade student as a buddy. No ninth grader is assigned more than one sixth-grade buddy. If $\frac{1}{3}$ of all the ninth graders are paired with $\frac{2}{5}$ of all the sixth graders, what fraction of the total number of sixth and ninth graders have a buddy?

A $\frac{2}{15}$

B $\frac{4}{11}$

C $\frac{11}{30}$

D $\frac{3}{8}$

E $\frac{11}{15}$

Solution(s):

First, let the number of sixth-grader be s and let the number of ninth-graders be n. Since they are paired in a way that everyone has one buddy, we know $\frac{1}{3}n = \frac{2}{5}s$. This means $5n = 6s$. Using algebra, we get

$$11n = 6(s+n)$$
$$11s = 5(s+n),$$

so we get

$$n = (s+n)\frac{6}{11}$$
$$s = (s+n)\frac{5}{11}.$$

Now, we must determine how many people have a buddy divided by the total number of people. There are $\frac{1}{3}n + \frac{2}{5}s$ people who have buddies and $s+n$ total people. Therefore:

$$\begin{aligned}\frac{\frac{1}{3}n+\frac{2}{5}s}{s+n} &= \frac{\frac{1}{3}\left(\frac{6}{11}(s+n)\right)}{s+n} \\ &+ \frac{\frac{2}{5}\left(\frac{5}{11}(s+n)\right)}{s+n} \\ &= \frac{2}{11}+\frac{2}{11} \\ &= \frac{4}{11}.\end{aligned}$$

Thus, the correct answer is **B**.

17. Jeremy's father drives him to school in rush hour traffic in 20 minutes. One day there is no traffic, so his father can drive him 18 miles per hour faster and gets him to school in 12 minutes. How far in miles is it to school?

A 4

B 6

C 8

D 9

E 12

Solution(s):

Let the speed in rush hour be s miles per hour. Since 20 minutes is $\frac{1}{3}$ hours, Jeremy's father drives $s(\frac{1}{3})$ miles during rush hour. When it's not rush hour, Jeremy's father drives at a speed of $s+18$ miles per hour. Since 12 minutes is $\frac{1}{5}$ hours, he drives $(s+18)(\frac{1}{5})$ miles when its not rush hour.

As the distance to school is the same, we know

$$\begin{aligned} s\frac{1}{3} &= (s+18)\frac{1}{5} \\ s\frac{5}{3} &= s+18 \\ s &= 27. \end{aligned}$$

Since the distance to school is $s(\frac{1}{3})$, we can deduce that the distance to school is $(27)(\frac{1}{3}) = 9$ miles.

Thus, the correct answer is **D**.

18. An arithmetic sequence is a sequence in which each term after the first is obtained by adding a constant to the previous term. For example, $2, 5, 8, 11, 14$ is an arithmetic sequence with five terms, in which the first term is 2 and the constant 3 is added. Each row and each column in this 5×5 array is an arithmetic sequence with five terms. What is the value of X?

1				25
		X		
17				81

- A 21
- B 31
- C 36
- D 40
- E 42

Solution(s):

1		13		25
		31		
17		49		81

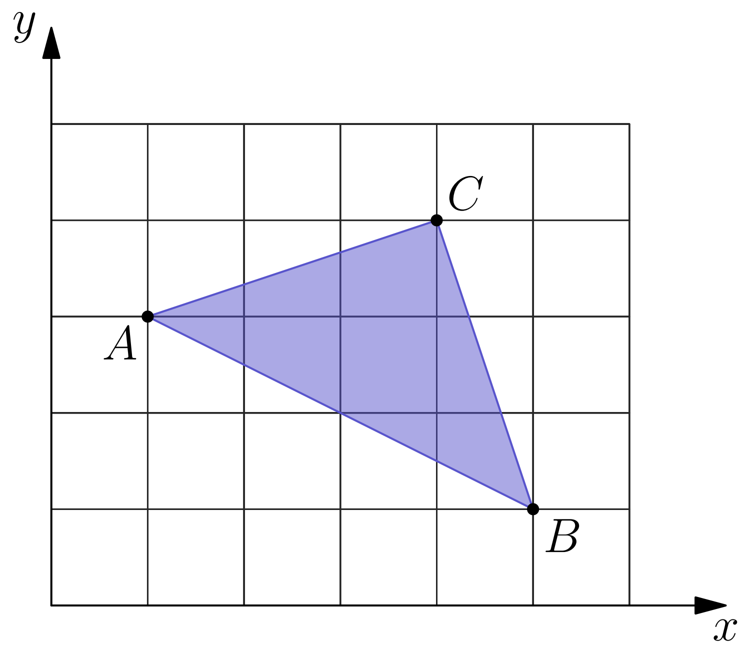
We can write an arbitrary 5-term arithmetic sequence as:

$$a, a+b, a+2b, a+3b, a+4b.$$

In this case, this representation of sequences implies that $a+2b = \dfrac{(a)+(a+4b)}{2}$, so the third team is the average of the first and last terms. Therefore, the middle number on the top row is the average of the top two corners $\dfrac{1+25}{2} = 13$, and the middle number on the bottom row is the average of the bottom two corners, which would be $\dfrac{17+81}{2} = 49$. Finally, the middle entry is the average of the third entry in the first and last row, which is $\dfrac{13+49}{2} = 31.$

Thus, the correct answer is **B**.

19. A triangle with vertices as $A = (1, 3)$, $B = (5, 1)$, and $C = (4, 4)$ is plotted on a 6×5 grid. What fraction of the grid is covered by the triangle?



A $\frac{1}{6}$

B $\frac{1}{5}$

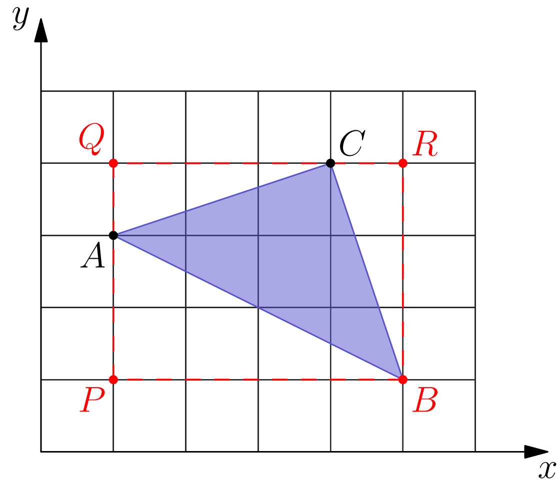
C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution(s):

The total area of the grid is $6 \cdot 5 = 30$. In order to find the fraction of this grid that the triangle covers, we must now find the area of the triangle. To do this, we will use the following diagram:



Thus, the area of the triangle $A(\triangle ABC)$ is equal to:

$$\begin{aligned}
&A(PQRB) - A(\triangle PAB) \\
&- A(\triangle BCR) - A(\triangle CAQ) \\
&= (3)(4) - \frac{1}{2}(4)(2) - 2\left(\frac{3}{2}\right) \\
&= 12 - 4 - 3 \\
&= 5.
\end{aligned}$$

Therefore, the fraction of the area is $\dfrac{5}{30} = \dfrac{1}{6}$.

Thus, the correct answer is **A**.

20. Ralph went to the store and bought 12 pairs of socks for a total of $24. Some of the socks he bought cost $1 a pair, some of the socks he bought cost $3 a pair, and some of the socks he bought cost $4 a pair. If he bought at least one pair of each type, how many pairs of $1 socks did Ralph buy?

A 4

B 5

C 6

D 7

E 8

Solution(s):

Let a be the number of $\$1$ pairs, let b be the number of $\$3$ pairs, and let c be the number of $\$4$ pairs of socks Ralph bought.

We know

$$a + b + c = 12$$

$$a + 3b + 4c = 24.$$

Therefore, we can subtract the two equations to get that

$$2b + 3c = 12.$$

Since $b > 0$, we know $2b > 0$ and therefore $12 - 3c > 0$. Therefore, $c < 4$.

Also, we know $c > 0$. Taking $\pmod 2$ of $2b + 3c = 12$ yields

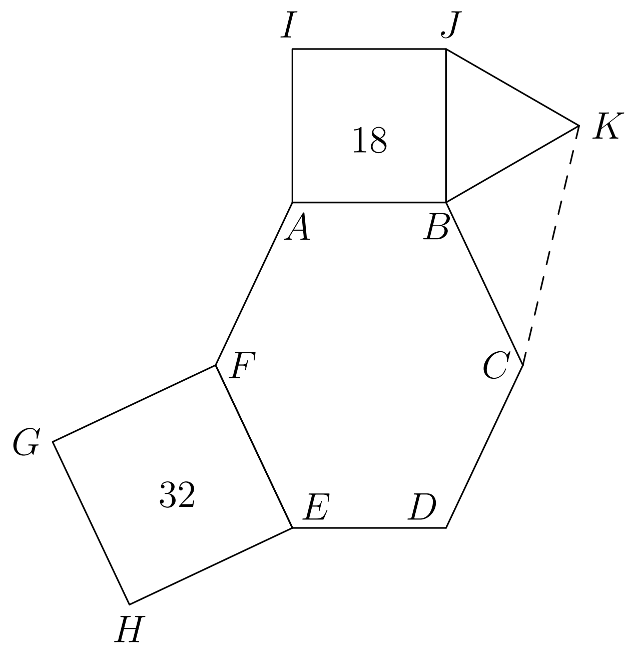
$$c \equiv 0 \pmod 2.$$

Since $c \equiv 0 \pmod 2$ and $0 < c < 4$, we know $c = 2$. Substituting, we deduce $b = 3$. Finally, with $a + b + c = 12$, we get $a + 2 + 3 = 12$, so $a = 7$.

This means the number of $\$1$ pairs is 7.

Thus, the correct answer is **D**.

21. In the given figure hexagon $ABCDEF$ is equiangular, $ABJI$ and $FEHG$ are squares with areas 18 and 32 respectively, $\triangle JBK$ is equilateral and $FE = BC$. What is the area of $\triangle KBC$?



A $6\sqrt{2}$

B 9

C 12

D $9\sqrt{2}$

E 32

Solution(s):

Since $ABJI$ is a square, each side has a length equal to the square root of the area. This means $JB = \sqrt{18} = 3\sqrt{2}$. Since JBK is equiangular, $\angle K = \angle J$. This means

$$BK = JB = 3\sqrt{2}.$$

Similarly, we get

$$EF = \sqrt{32} = 4\sqrt{2}.$$

Since $ABCDEF$ is equiangular, we can get that

$$BC = EF = 4\sqrt{2}.$$

Therefore, the area of KBC is equal to

$$\begin{aligned} \frac{(BC)(BK)}{2} &= \frac{(4\sqrt{2})(3\sqrt{2})}{2} \\ &= \frac{24}{2} \\ &= 12. \end{aligned}$$

Thus, the correct answer is **C**.

22. On June 1, a group of students is standing in rows, with 15 students in each row.

On June 2, the same group is standing with all of the students in one long row.

On June 3, the same group is standing with just one student in each row.

On June 4, the same group is standing with 6 students in each row.

This process continues through June 12 with a different number of students per row each day. However, on June 13, they cannot find a new way of organizing the students. What is the smallest possible number of students in the group?

A 21

B 30

C 60

D 90

E 1080

Solution(s):

Let n be the number of students. Since we consider only 12 days, and each day's row and column pair represents a unique factor pair of n, we have 12 factors of n.

Since 15 students were in one row, n is a multiple of 15.

Since 6 students were in one row on a different day, n is a multiple of 6.

Since n is a multiple of 6 and 15, it must be a multiple of $lcm(6, 15) = 30$. This means n is a multiple of $2 \cdot 3 \cdot 5$. The number of factors is the product of the exponents plus 1, so this would have 8 factors. However, multiplying it by 2 yields $2^2 \cdot 3 \cdot 5 = 60$. This yields 12 total factors as required.

Thus, the correct answer is **C**.

23. Tom has twelve slips of paper which he wants to put into five cups labeled A, B, C, D, E.

He wants the sum of the numbers on the slips in each cup to be an integer. Furthermore, he wants the five integers to be consecutive and increasing from A to E. The numbers on the papers are:

$$2, 2, 2, 2.5, 2.5, 3,$$
$$3, 3, 3, 3.5, 4, 4.5.$$

If a slip with 2 goes into cup E and a slip with 3 goes into cup B, then the slip with 3.5 must go into what cup?

- A A
- B B
- C C
- D D
- E E

Solution(s):

The sum of every slip combined is 35. Therefore, the average of the cups would be 7. This would make

$$A = 5$$
$$B = 6$$
$$C = 7$$
$$D = 8$$
$$E = 9.$$

Since B has one 3, the sum of the remaining slips is 3, which can only be done with another 3.

As such, what remains after filling B is:

$$2, 2, 2, 2.5, 2.5, 3,$$
$$3, 3.5, 4, 4.5.$$

Let E' be the sum of E after putting the 2 slip. This equals $9-2=7$.

Therefore, we need to fill C, E' each to be 7 and have A have 5 and D having 8. Now, lets try fitting the 3.5 slip into each of the cups.

When putting it in A, we have 1.5 left which is invalid. Similarly, we can't put it in B as it is full. When putting it in C, we have 3.5 left, which cannot work as no other values sum to 3.5. Similar to C, putting it in E would have 3.5, which we just showed to be invalid. Finally, we can make cup D work by making the cups be

$$\begin{aligned}&\{2.5, 2.5\}, \{3, 3\}, \{3, 4\},\\ &\{3.5, 4.5\}, \{2, 2, 2, 3\}.\end{aligned}$$

Thus, the correct answer is **D**.

24. A baseball league consists of two four-team divisions. Each team plays every other team in its division N games. Each team plays every team in the other division M games with $N > 2M$ and $M > 4$. Each team plays a 76 game schedule.

How many games does a team play within its own division?

- A 36
- B 48
- C 54
- D 60
- E 72

Solution(s):

Since there are 3 other teams in the division and 4 teams not in the division, a team plays $3N + 4M = 76$ teams. Also notice that

$$76 = 3N + 4M > 10M,$$

since $N > 2M$. This implies that $76 > 10M$, and as such, $7 \geq M$.

Under the given conditions, we know

$$7 \geq M > 4.$$

Also, taking

$$3N + 4N = 76$$

under modulus 3 yields

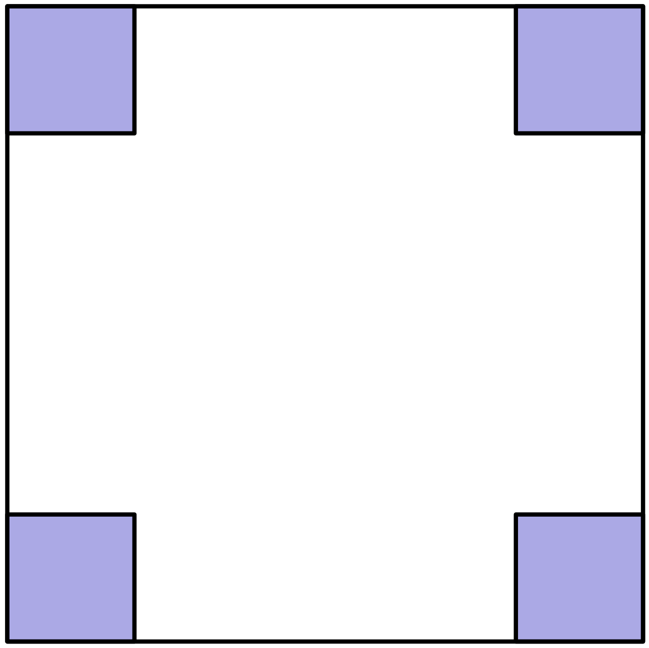
$$M \equiv 1 \pmod{3}.$$

The only possible M in the range is 7, suggesting that he plays $4 \cdot 7 = 28$ games against non-divisional opponents.

As such, he plays $76 - 28 = 48$ games against divisional opponents.

Thus, the correct answer is **B**.

25. One-inch squares are cut from the corners of this 5 inch square. What is the area in square inches of the largest square that can be fitted into the remaining space?



A 9

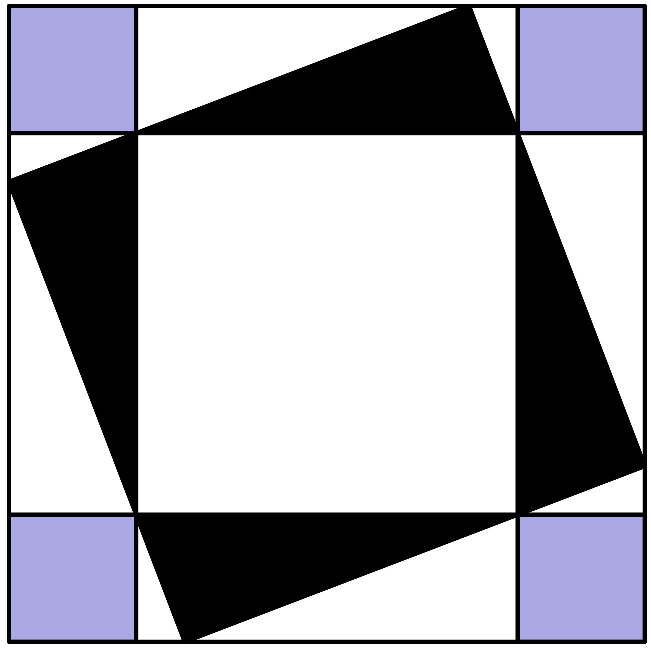
B $12\frac{1}{2}$

C 15

D $15\frac{1}{2}$

E 17

Solution(s):



The desired square can be drawn like this. It consists a smaller square of size 3×3 in it, with area $3 \cdot 3 = 9$, as well as 4 triangles. As each square has base 3 and height 1, the combined area of the triangles is $4(\frac{3 \cdot 1}{2}) = 6$.

This makes the total area of this square equal to 15.

Thus, the correct answer is **C**.

Problems: https://live.poshenloh.com/past-contests/amc8/2015