2026 AMC 12B Solutions

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1. A souvenir shop sells solid metal sculptures of Achilles in two sizes. One is 6 cm tall and one is 8 cm tall. Proportions between the two sculptures are preserved. The smaller sculpture has a mass of 486 grams. What is the mass, in grams, of the larger sculpture?

A 648

B 864

C 972

D 1152

E 1458

Small hint:

Mass scales with volume

Big hint:

The length scale factor is $\frac{4}{3}$

Solution:

The volume, and hence mass, scales by the cube of the length ratio. The larger mass is

$$\begin{aligned} 486 \times \left(\frac{4}{3}\right)^3 &= 18 \times 64 \\ &= 1152. \end{aligned}$$

Thus, the correct answer is **D**.

2. Bees occupy the 8 corners of a $1 \times 2 \times 3$ rectangular box. The bees can fly inside the box, but cannot fly farther than $\frac{1}{2}$ unit from their respective corners. What is the volume, in cubic units, of the region inside the box that cannot be reached by any bee?

A $6 - \frac{\pi}{3}$

B $6 - \frac{\pi}{4}$

C $6 - \frac{\pi}{6}$

D $6 - \frac{\pi}{8}$

E $6 - \frac{\pi}{12}$

Small hint:

Each bee reaches one eighth of a sphere

Big hint:

The eight reachable regions have no overlapping volume

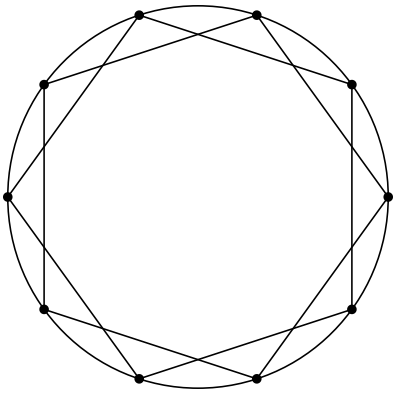
Solution:

The eight spherical eighths together have the volume of one sphere of radius $\frac{1}{2}$. Subtracting from the box gives

$$\begin{aligned} 1 \times 2 \times 3 - \frac{4}{3}\pi \left(\frac{1}{2}\right)^3 \\ = 6 - \frac{\pi}{6}. \end{aligned}$$

Thus, the correct answer is **C**.

3. There are 2026 equally spaced points that all lie on a circle. A line segment is drawn between two points if there is exactly one point between them along the circle. Into how many regions will this divide the interior of the circle? The figure below shows an example with 10 equally spaced points, in which case there are 21 interior regions.



A 4052

B 4053

C 4054

D 6079

E 6081

Small hint:

Count the central region, the small triangles, and the regions along the circle separately

Big hint:

Match the small triangles to marked points and the outer regions to arcs between consecutive points

Solution:

There is 1 central region. Each marked point is the tip of one small triangle, giving 2026 triangular regions. Each arc between consecutive marked points borders one outer region, giving another 2026. The total is

$$1 + 2026 + 2026 = 4053.$$

Alternatively, use Euler's formula for a connected planar graph: $V - E + F = 2$, where V counts vertices, E counts edges, and F counts faces, including the region outside the circle.

Each of the 2026 segments crosses exactly two others. Each crossing is counted from both segments, so there are 2026 interior vertices. Add the marked points to get V. Each segment is cut into three edges, and the circle contributes 2026 arcs. Thus

$$V = 2026 + 2026 = 4052$$
$$E = 3 \times 2026 + 2026 = 8104$$

Euler's formula gives $F = 2 - 4052 + 8104 = 4054$. Subtract the outside face to get 4053 regions inside the circle.

Thus, the correct answer is **B**.

4. For how many integers n is

$$\frac{n^2-1}{n+2}$$

also an integer?

A 2

B 3

C 4

D 6

E infinitely many

Small hint:

Divide n^2-1 by $n+2$ with remainder

Big hint:

$n^2-1=(n+2)(n-2)+3$

Solution:

The expression equals $n-2+\frac{3}{n+2}$. It is an integer exactly when $n+2$ is one of -3, -1, 1, and 3. These give four distinct integers n, none equal to -2.

Thus, the correct answer is **C**.

5. Let $a = \log_2 27$, let $b = \log_3 25$, and let $c = \log_{32} 5$. What is $\frac{ab}{c}$?

A 25

B 27

C 30

D 32

E 36

Small hint:

Write every logarithm in the same base

Big hint:

$27 = 3^3, 25 = 5^2$, and $32 = 2^5$

Solution:

Changing to base 2, we have

$$a = 3 \log_2 3$$
$$b = \frac{2 \log_2 5}{\log_2 3}$$
$$c = \frac{\log_2 5}{5}$$

The logarithms cancel in $\frac{ab}{c}$, leaving $3 \times 2 \times 5 = 30$.

Thus, the correct answer is **C**.

6. The integer n satisfies the inequalities

$$\begin{aligned}\frac{n}{100} &< \frac{1}{5\cdot 10}\\ +\frac{1}{10\cdot 15}&+\frac{1}{15\cdot 20}\\ &+\cdots+\frac{1}{90\cdot 95}\\ +\frac{1}{95\cdot 100} &< \frac{n+1}{100}.\end{aligned}$$

What is n?

A 2

B 3

C 4

D 5

E 6

Small hint:

$\frac{1}{k(k+5)} = \frac{1}{5}(\frac{1}{k} - \frac{1}{k+5})$

Big hint:

After splitting each fraction, the negative parts cancel the next positive parts

Solution:

The two factors in each denominator differ by 5. For a denominator $k(k+5)$, we have

$$\begin{aligned}\frac{1}{k}-\frac{1}{k+5} &= \frac{(k+5)-k}{k(k+5)} \\ &= \frac{5}{k(k+5)}.\end{aligned}$$

Each original fraction is therefore one fifth of such a difference.

Let S be the sum in the question. Splitting its terms this way gives

$$S=\frac{1}{5}\left[\begin{array}{l}\left(\frac{1}{5}-\frac{1}{10}\right) \\ +\left(\frac{1}{10}-\frac{1}{15}\right) \\ +\left(\frac{1}{15}-\frac{1}{20}\right) \\ +\cdots \\ +\left(\frac{1}{95}-\frac{1}{100}\right)\end{array}\right].$$

Inside the brackets, the first $-\frac{1}{10}$ cancels the next $+\frac{1}{10}$, since they add to zero. The same happens to $-\frac{1}{15}$ and $+\frac{1}{15}$, and the cancellation continues down the list. Only the first $\frac{1}{5}$ and the last $-\frac{1}{100}$ remain.

Thus

$$S=\frac{1}{5}\left(\frac{1}{5}-\frac{1}{100}\right)=\frac{19}{500}.$$

Multiplying the original inequalities by 100 gives $n<\frac{19}{5}<n+1$. Since $3<\frac{19}{5}<4$, we have $n=3$.

Thus, the correct answer is **B**.

7. Sharon tosses a fair coin six times and keeps track of how many heads and tails she has gotten. Her first toss is heads. What is the probability that after every toss Sharon's count of heads differs from her count of tails?

Express your answer as a common fraction.

A $\frac{1}{8}$

B $\frac{3}{16}$

C $\frac{1}{4}$

D $\frac{5}{16}$

E $\frac{3}{8}$

Small hint:

The second toss must also be heads

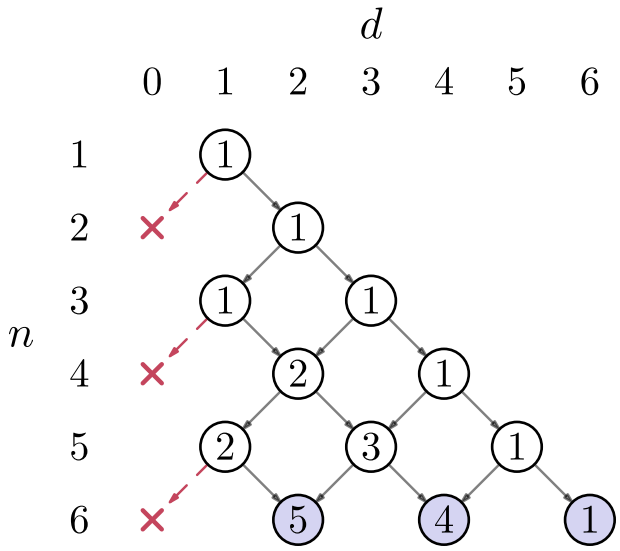
Big hint:

Count sequences that start with heads and never tie. At lead d, add the previous counts for $d-1$ and $d+1$. Divide the last row's total by 2^5.

Solution:

After the first head, heads must stay ahead of tails: a negative lead would require passing through a tie.

In the diagram, n is the number of tosses and d is heads minus tails. A circled number counts sequences that have avoided a tie. Start with 1 after the given first head. A head moves one column right and a tail one column left. Cross out every route that reaches $d = 0$.



To fill a circle, add the numbers at the starts of its incoming arrows. For example, the 3 in row 5 is $2+1$, and the 5 in row 6 is $2+3$.

The last row has $5+4+1=10$ successful sequences. The first toss is fixed, so the remaining five tosses have $2^5=32$ equally likely outcomes. Hence

$$\frac{10}{32}=\frac{5}{16}.$$

Thus, the correct answer is **D**.

8. A regular octahedron has 8 equilateral triangle faces with edges of length $\sqrt{2}$. A plane is perpendicular to the line segment joining two opposite vertices of the octahedron and intersects that line segment $\frac{1}{2}$ unit from one of its endpoints. What is the area of the intersection of that plane and the octahedron?

Express your answer as a common fraction.

A $\frac{1}{4}$

B $\frac{1}{2}$

C $\frac{1}{2}\sqrt{2}$

D $\frac{3}{4}$

E $\sqrt{2}$

Small hint:

View the octahedron as two square pyramids joined at their bases

Big hint:

The altitude of each pyramid is 1

Solution:

The central square has side $\sqrt{2}$ and area 2. Its center is distance 1 from each base vertex, so a lateral edge of length $\sqrt{2}$ gives pyramid altitude 1. A parallel cross-section halfway from the apex to the base has half the side length and one quarter of the area, or $\frac{1}{2}$.

Thus, the correct answer is **B**.

9. Suppose that a and b are positive integers satisfying $20a + 26b = 2026$. What is the least possible value of $a + b$?

A 71

B 80

C 83

D 86

E 88

Small hint:

Minimize the count by using as many 26s as possible

Big hint:

$10a + 13b = 1013$ implies $b \equiv 1 \pmod{10}$

Solution:

The reduced equation is $10a + 13b = 1013$. Reducing modulo 10 gives $3b \equiv 3 \pmod{10}$, hence $b \equiv 1 \pmod{10}$. Also

$$a + b = \frac{1013 - 3b}{10},$$

so maximize b. Positivity of a forces $b \leq 77$, making $b = 71$ the largest eligible value. Then $a = 9$, giving $a + b = 80$.

Thus, the correct answer is **B**.

10. Point C lies on a circle with diameter $\overline{AB}$. Point D lies on $\overline{AB}$, 14 units from A, 24 units from B, and 16 units from C. What is the area, in square units, of $\triangle ABC$?

A $186\sqrt{2}$

B $152\sqrt{3}$

C 264

D $187\sqrt{2}$

E $108\sqrt{6}$

Small hint:

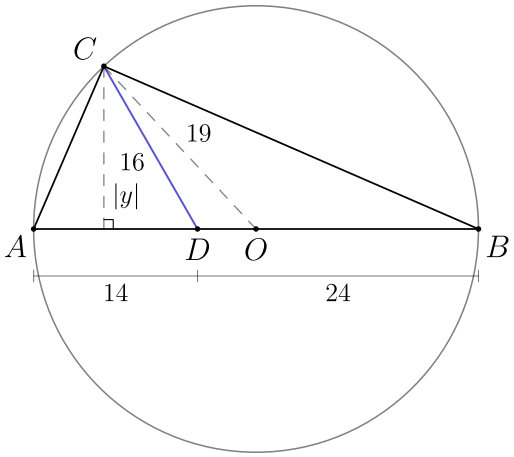
Let the center of the circle be O; then $OD = 5$ and $OC = 19$

Big hint:

Use coordinates with $D = (0, 0)$ and $O = (5, 0)$

Solution:

The circle has radius 19 and center O, with $OD = 5$. Put $D = (0, 0)$ and $O = (5, 0)$. Write $C = (x, y)$. The dashed perpendicular shows the height of the triangle.



Then

$$x^2 + y^2 = 256$$
$$(x-5)^2 + y^2 = 361$$

Subtracting gives $-10x + 25 = 105$, so $x = -8$ and $|y| = \sqrt{256 - 64} = 8\sqrt{3}$. The area is $\frac{1}{2} \times 38 \times 8\sqrt{3} = 152\sqrt{3}$.

Thus, the correct answer is **B**.

11. A list contains 2026 statements, each of which is either true or false. The first statement is "The number of false statements in this list is at least 1001." The second statement is "The number of false statements in this list is at least 1002." In general, the kth statement in the list is "The number of false statements in this list is at least $k + 1000$." How many statements in the list are true?

A 342

B 513

C 1013

D 1026

E 1513

Small hint:

All true statements come before all false statements

Big hint:

If F statements are false, the true ones have indices $k \leq F - 1000$

Solution:

Let F be the number of false statements. If $F \leq 1000$, all statements would be false, contradicting that bound. Thus $F > 1000$, and there are exactly $F - 1000$ true statements. Consequently

$$F + (F - 1000) = 2026,$$

so $F = 1513$ and the number of true statements is $2026 - 1513 = 513$.

Thus, the correct answer is **B**.

12. Polly and Quentin are dancing. They start 1 meter apart. Polly rotates 120° clockwise around Quentin while he remains still, staying 1 meter away throughout. Then Quentin rotates 120° clockwise around Polly while she remains still, staying 1 meter away throughout. They repeatedly do this pair of moves from their new positions. How many meters from where she started is Polly after they have each moved 10 times?

A 0

B $\frac{\sqrt{3}}{2}$

C 1

D $\sqrt{3}$

E 2

Small hint:

A regular hexagon's 120° angles match the dancers' turns

Big hint:

Three pairs of moves return both dancers to their starting vertices

Solution:

Draw a regular hexagon $ABCDEF$ with side length 1 meter, labeling the vertices counterclockwise. Start Polly at A and Quentin at B.

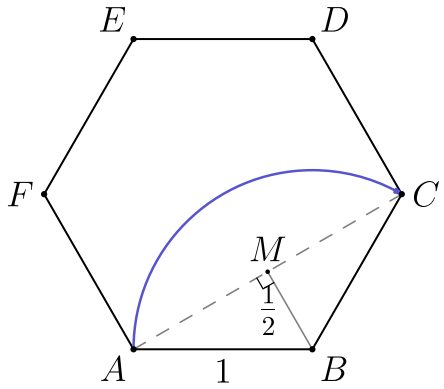
Each interior angle is 120°. Since $AB = BC = 1$, Polly's clockwise turn around B takes her from A to C. Quentin's turn around C then takes him from B to D.

Continuing the same way gives these positions after each pair of moves:

Pairs	Polly	Quentin
0	A	B
1	C	D
2	E	F
3	A	B

After three pairs, both are back at their starts. The same motion repeats from there. Since $10 = 3 \times 3 + 1$, Polly ends at C, just as after the first pair. Her distance from her start is AC.

Let M be the midpoint of $\overline{AC}$. In isosceles triangle ABC, BM is perpendicular to AC and bisects the 120° angle at B. Each right triangle has angles 30°, 60°, and 90°. Its hypotenuse is 1, so $BM = \frac{1}{2}$.



The arc shows Polly's first turn. The Pythagorean Theorem gives

$$\begin{aligned} AC &= 2AM \\ &= 2\sqrt{1 - \left(\frac{1}{2}\right)^2} \\ &= \sqrt{3}. \end{aligned}$$

Thus, the correct answer is **D**.

13. A jar contains 12 green marbles and 8 red marbles. In a game, Anika and Ben take turns drawing a random marble from the jar, with replacement. If Ben ever draws a red marble, he wins, and if Anika ever draws a green marble, she wins. Otherwise, the game continues with the next player's turn. The probability that Ben eventually wins is A if Anika goes first and B if Ben goes first. What is $B - A$?

Express your answer as a common fraction.

A $\frac{2}{25}$

B $\frac{4}{25}$

C $\frac{4}{19}$

D $\frac{6}{19}$

E $\frac{9}{20}$

Small hint:

$A = \frac{2}{5}B$

Big hint:

$B = \frac{2}{5} + \frac{3}{5}A$

Solution:

Anika must first draw red for Ben to win when she starts, so $A = \frac{2}{5}B$. When Ben starts, he either wins immediately or draws green, giving

$$B = \frac{2}{5} + \frac{3}{5}A = \frac{2}{5} + \frac{6}{25}B.$$

Hence $B = \frac{10}{19}$, $A = \frac{4}{19}$, and $B - A = \frac{6}{19}$.

Thus, the correct answer is **D**.

14. A line segment is drawn between two points on the graph of $y = x^2$ whose x-coordinates have been chosen independently and uniformly at random in the interval $[0, 2]$. The probability that the y-coordinate of the midpoint of the segment is greater than 2 lies in which of the following intervals?

A $(0.20, 0.21)$

B $(0.21, 0.22)$

C $(0.22, 0.23)$

D $(0.23, 0.24)$

E $(0.24, 0.25)$

Small hint:

For chosen x-coordinates u and v, the condition is $u^2 + v^2 > 4$

Big hint:

Subtract a quarter disk from a 2-by-2 square

Solution:

Let u and v be the chosen x-coordinates. The sample space is $[0, 2]^2$. The midpoint condition $\frac{u^2+v^2}{2} > 2$ selects the part outside a quarter disk of radius 2. Hence

$$P = \frac{4 - \pi}{4} = 1 - \frac{\pi}{4}.$$

The familiar bounds $3.14 < \pi < \frac{22}{7}$ give $\frac{3}{14} < P < 0.215$. Since $\frac{3}{14} > 0.21$, the probability is in $(0.21, 0.22)$.

Thus, the correct answer is **B**.

15. What is the value of

$$\log_2\left(\sin\frac{\pi}{8}\right) + \log_2\left(\sin\frac{3\pi}{8}\right) + \log_2\left(\sin\frac{5\pi}{8}\right) + \log_2\left(\sin\frac{7\pi}{8}\right)?$$

A -3

B $-2\sqrt{2}$

C -2

D $-\frac{3}{2}$

E $-\sqrt{2}$

Small hint:

Combine the logarithms into the logarithm of a product

Big hint:

$\sin\frac{5\pi}{8} = \sin\frac{3\pi}{8}$ and $\sin\frac{7\pi}{8} = \sin\frac{\pi}{8}$

Solution:

The product inside the combined logarithm is

$$\begin{aligned} &\left(\sin\frac{\pi}{8}\sin\frac{3\pi}{8}\right)^2 \\ =\ &\left(\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2 \\ =\ &\left(\frac{1}{2}\sin\frac{\pi}{4}\right)^2 = \frac{1}{8}. \end{aligned}$$

Thus the sum is $\log_2 \frac{1}{8} = -3.$

Thus, the correct answer is **A**.

16. Annie, Bobby, and Casey each receive an integer, not necessarily distinct, from 1 through 10. In how many ways can the numbers be given so that their sum is divisible by 7?

A 139

B 140

C 142

D 143

E 145

Small hint:

Dividing $8, 9,$ and 10 by 7 leaves remainders $1, 2,$ and 3

Big hint:

Among 1 through 7, exactly one number completes any given sum to a multiple of 7

Solution:

Here a residue modulo 7 means the remainder after division by 7. The numbers 1 through 7 give every remainder once, so one of them always completes any two chosen numbers to a multiple of 7.

Count by how many people receive a number in 8 through 10. If none do, the first two numbers have 7^2 choices, and the third has exactly one suitable residue: 49 ways. If exactly one does, choose that person and their number, then one of the other numbers; the last is determined: $3 \times 3 \times 7 = 63$ ways.

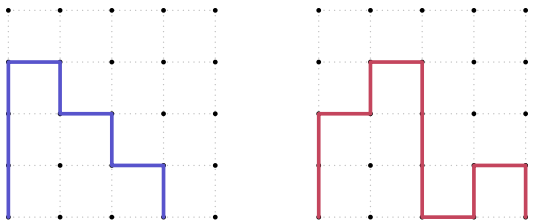
If exactly two do, choose the person who receives a number from 1 through 7 in 3 ways. Each of the other two people has 3 choices, with residues in $\{1, 2, 3\}$. The remaining number is determined: $3 \times 3^2 = 27$ ways. If all three do, their residue sum must be 7, giving permutations of $(1, 3, 3)$ and $(2, 2, 3)$, or 6 ways. The total is $49 + 63 + 27 + 6 = 145$.

Thus, the correct answer is **E**.

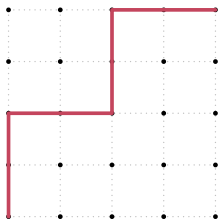
17. Belinda follows a path on the plane that is entirely contained within the square region $0 \leq x \leq 4$ and $0 \leq y \leq 4$. Call her path *dignified* if and only if all of the following conditions are satisfied.

- Each step is one unit up, down, or to the right.
- The path does not revisit any vertex.
- The path starts at $(a, 0)$ and ends at $(b, 0)$, where a and b are integers with $a < b$, and touches the x-axis only at these two points.

The following figure shows some examples of dignified and not dignified paths.


Dignified

Not dignified


Not dignified

How many dignified paths can Belinda follow?

A 174

B 340

C 448

D 512

E 970

Small hint:

Count by the horizontal distance $d = b - a$

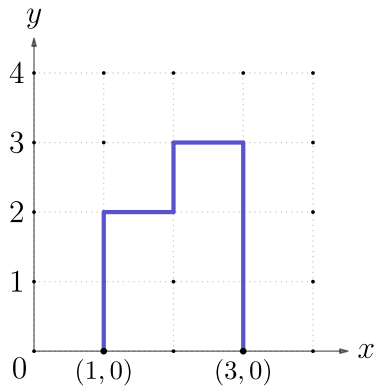
Big hint:

Choose the height of each rightward step; the vertical parts are then forced

Solution:

Fix the number of rightward steps. Each must be at height $1, 2, 3,$ or $4,$ because the path stays in the square and touches the x-axis only at its endpoints.

For example, take two rightward steps at heights 2 and $3,$ starting at $(1, 0).$ The path must go up to height $2,$ right, up to height $3,$ right, then down to the axis.



Once the start and heights are chosen, the vertical parts are forced. Between rightward steps, go straight up or down to the next height. Reversing direction would revisit a vertex. Every choice of heights gives a valid path: the heights stay positive, and after moving right the path never returns to an earlier column.

With two rightward steps, the starting x-coordinate can be $0, 1,$ or 2 : there are 3 starting positions. Each step independently has 4 height choices, so this case contributes $3 \times 4^2 = 48$ paths.

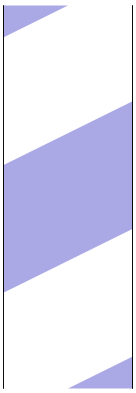
In general, let $d = b - a$ be the number of rightward steps. It can be $1, 2, 3,$ or $4.$ The starting x-coordinate can be any integer from 0 through $4 - d,$ giving $5 - d$ positions. There are 4^d choices for the heights.

The total number of paths is

$$\begin{aligned} &4 \times 4 + 3 \times 4^2 \\ &\quad + 2 \times 4^3 + 4^4 \\ &= 448. \end{aligned}$$

Thus, the correct answer is **C**.

18. A ribbon wraps around a pole in such a way that equal amounts of area are covered and not covered by the ribbon. The width of the ribbon is 4 cm, and the circumference of the pole is 10 cm. The figure below shows a possible front view of a portion of the pole.



In centimeters, how much of the pole height is needed for the ribbon to wrap around exactly once?

Express your answer as a common fraction.

A $\frac{15}{2}$

B 8

C 10

D $\frac{25}{2}$

E $\frac{40}{3}$

Small hint:

Unroll the cylinder into a plane

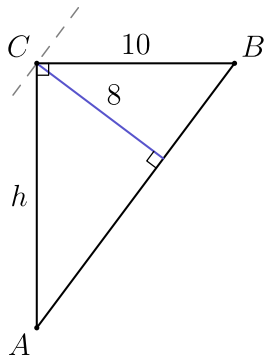
Big hint:

The perpendicular distance between matching edges of successive turns is 8 cm

Solution:

Unroll the pole. The ribbon becomes parallel strips of width 4, with gaps of the same width. Matching edges of consecutive strips are therefore 8 apart, measured perpendicular to the strips.

Let h be the height gained in one turn. In the unrolled surface, draw a right triangle with vertical leg $AC = h$ and horizontal leg $CB = 10$. Its sloping side AB follows a ribbon edge. The next matching edge passes through C parallel to AB, so the perpendicular from C to AB has length 8.



Using the two different bases of triangle ABC, its area is both $\frac{1}{2} \times 10 \times h$ and $\frac{1}{2} \times 8 \times \sqrt{100 + h^2}$. Equating these and squaring gives

$$\begin{aligned} 100h^2 &= 64(100 + h^2) \\ 36h^2 &= 6400 \\ h &= \frac{40}{3} \end{aligned}$$

Thus, the correct answer is **E**.

19. Starting with a list of three numbers, form a sequence in which each subsequent term is the average of the previous three terms. For example, starting with the sequence $(2, 4, 9)$, the next two terms are $\frac{2+4+9}{3} = 5$ and $\frac{4+9+5}{3} = 6$. Which of the following initial three terms will produce a sequence that has the greatest 2026th term?

A $(1, 2, 3)$

B $(2, 1, 1)$

C $(3, 4, 8)$

D $(4, 4, 7)$

E $(4, 5, 6)$

Small hint:

Compare corresponding terms by subtracting the sequences

Big hint:

Once three consecutive differences are positive, every later difference is positive

Solution:

Option C is larger term by term than A and B. Subtract D from C: the first six differences are

$$-1,\ 0,\ 1,\ 0,\ \frac{1}{3},\ \frac{4}{9}.$$

The next difference is positive, so from the fifth term onward every difference is positive. Subtract E from C: the first six differences are

$$-1,\ -1,\ 2,\ 0,\ \frac{1}{3},\ \frac{7}{9}.$$

The same argument applies. Thus C has the largest term at every index from the fifth onward, including 2026.

Thus, the correct answer is **C**.

20. Which of the following statements most accurately describes the graph of the relationship

$$\sin(x^2) = \cos(y^2)?$$

A the set of all points $\left(\sqrt{\frac{m}{\pi}}, \sqrt{\frac{n+\frac{1}{2}}{\pi}}\right)$, where m and n are nonnegative integers

B the set of all lines $y = \pm x + \frac{\pi}{2} + k\pi$, where k is an integer

C the set of all circles $x^2 + y^2 = \frac{\pi}{2} + 2\pi n$, where n is a nonnegative integer

D a set of infinitely many concentric circles and infinitely many hyperbolas

E a set of finitely many ellipses centered at the origin and infinitely many parabolas

Small hint:

$\cos(y^2) = \sin(\frac{\pi}{2} - y^2)$

Big hint:

$\sin u = \sin v$ when $u = v + 2k\pi$ or $u = \pi - v + 2k\pi$

Solution:

The sine identity gives the two families

$$x^2 + y^2 = \frac{\pi}{2} + 2k\pi$$
$$x^2 - y^2 = \frac{\pi}{2} + 2k\pi$$

In the first family $k \geq 0$ gives concentric circles. In the second, every integer k gives a nondegenerate hyperbola, since the right side is never zero. Both families are infinite.

Thus, the correct answer is **D**.

21. Let T_n be the nth triangular number, that is, $T_1 = 1, T_2 = 1 + 2 = 3, T_3 = 1 + 2 + 3 = 6$, and so on. The value of

$$\sum_{k=1}^{\infty} T_k \cdot \left(\frac{1}{3}\right)^k$$

can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

A 14

B 15

C 16

D 17

E 18

Small hint:

Subtract one third of the series from itself

Big hint:

$T_k - T_{k-1} = k$

Solution:

Let

$$S = \sum_{k=1}^{\infty} \frac{T_k}{3^k}.$$

Dividing by 3 multiplies each denominator by 3, so

$$\begin{aligned} S &= \frac{1}{3} + \frac{3}{3^2} + \frac{6}{3^3} + \cdots \\ \frac{S}{3} &= \qquad \frac{1}{3^2} + \frac{3}{3^3} + \cdots \end{aligned}$$

Subtract the second row from the first, matching equal denominators. The first numerator stays 1. The next numerators are $3-1=2, 6-3=3$, and so on, because consecutive triangular numbers differ by k. Define

$$U=\sum_{k=1}^{\infty}\frac{k}{3^k}.$$

Thus $S-\frac{S}{3}=U$, or $\frac{2}{3}S=U$. Repeat the subtraction with the new series:

$$\begin{aligned} U &= \frac{1}{3}+\frac{2}{3^2}+\frac{3}{3^3}+\cdots \\ \frac{U}{3} &= \qquad \frac{1}{3^2}+\frac{2}{3^3}+\cdots \end{aligned}$$

Now the numerators are $1, 2-1=1, 3-2=1$, and so on. Hence

$$\frac{2}{3}U=\sum_{k=1}^{\infty}\frac{1}{3^k}.$$

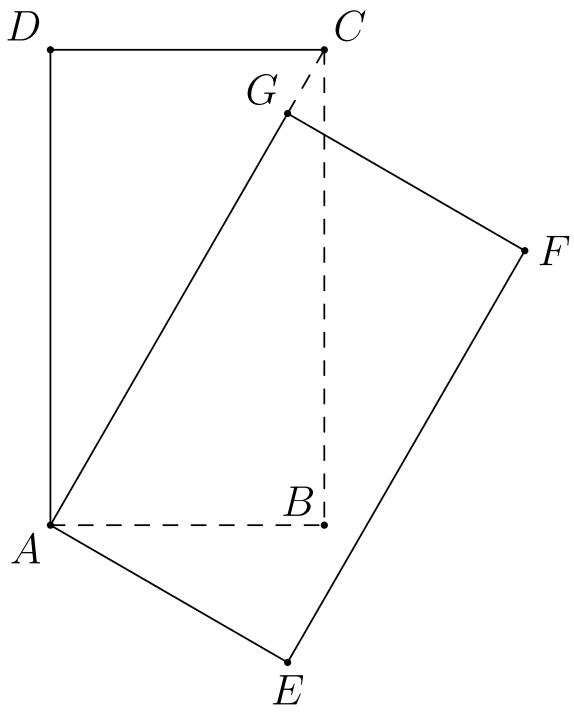
This geometric series has first term $\frac{1}{3}$ and common ratio $\frac{1}{3}$, so

$$\frac{2}{3}U=\frac{\frac{1}{3}}{1-\frac{1}{3}}=\frac{1}{2}.$$

Therefore $U=\frac{3}{4}$ and $S=\frac{3}{2}U=\frac{9}{8}$. Thus $p=9, q=8$, and $p+q=17$.

Thus, the correct answer is **D**.

22. Rectangular card $ABCD$ has $AB = 1$ and $AC = 2$. As shown in the figure, a second card, $AEFG$, congruent to the first, is stacked onto it so that $\overline{AG}$ lies on $\overline{AC}$ and the two cards share corner A.



Additional cards are added, following the same pattern (with shared corner A), creating a fan of cards. The process stops when a new card fits exactly atop $ABCD$. The shape then created is a nonconvex polygon with rotational symmetry. What is the area of this polygon?

A $20\sqrt{3} - 24$

B $34\sqrt{3} - 48$

C $6 + 3\sqrt{3}$

D 12

E $8\sqrt{3}$

Small hint:

Each new card is rotated 30°, so there are 12 distinct cards

Big hint:

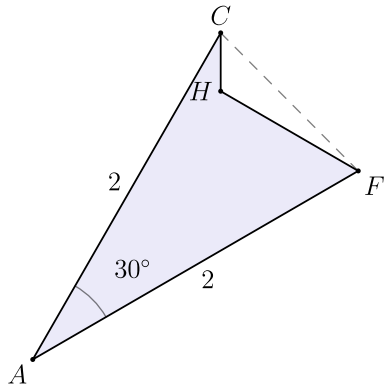
Find the boundary between two consecutive outer corners, including its inward notch

Solution:

The rectangle has sides 1 and $\sqrt{3}$, so the rotation is 30°. Place $A = (0,0), B = (1,0)$, and $C = (1,\sqrt{3})$. The next card's outer corner is $F = (\sqrt{3},1)$. Its upper edge has equation $x + \sqrt{3}y = 2\sqrt{3}$. It meets $x = 1$ at $H = (1, 2 - \frac{1}{\sqrt{3}})$.

Between the rays AF and AC, the boundary is FHC. Only the two pictured cards and the card rotated 30° counterclockwise from $ABCD$ meet the interior of this sector. In that third card, each ray makes at most 30° with the short side, so it reaches at most $\frac{1}{\cos 30^\circ} = \frac{2}{\sqrt{3}}$. The card ending at F reaches at least $\sqrt{3}$ along each ray. So the third card adds nothing here.

The area of triangle AFC is $\frac{1}{2} \times 2 \times 2\sin 30^\circ = 1$. The inward notch is triangle FHC. Its vertical base has length $HC = \sqrt{3} - (2 - \frac{1}{\sqrt{3}})$, and its horizontal height is $\sqrt{3} - 1$.



Subtract the notch from the large triangle to get the area of one sector:

$$1 - \frac{1}{2}\left(\frac{4}{\sqrt{3}} - 2\right)(\sqrt{3} - 1)$$
$$= \frac{5\sqrt{3}}{3} - 2.$$

There are 12 congruent sectors, giving $20\sqrt{3} - 24$.

Thus, the correct answer is **A**.

23. For positive integers a and b, let $R(a,b)$ denote the remainder when a is divided by b. Suppose N is the least positive integer such that

$$\begin{aligned} R(N,2) &\geq 1 \\ R(N,4) &\geq 2 \\ R(N,8) &\geq 4 \\ &\vdots \\ R(N,2^{200}) &\geq 2^{199} \end{aligned}$$

What are the last three digits of N?

A 125

B 249

C 375

D 511

E 875

Small hint:

The condition modulo 2^k forces the kth binary digit from the right to be 1

Big hint:

The smallest eligible integer is $2^{200}-1$

Solution:

The first condition makes N odd, so its remainder modulo 2 is 1. With divisor 4, the remainder must then be 3; with divisor 8, it must be 7. Each time we double the divisor, the new remainder is either the old remainder or the old remainder plus the old divisor. The lower bound forces the latter. Thus the remainder modulo 2^{200} is $2^{200}-1$, making the smallest possible value $N = 2^{200}-1$. To find its last three digits, work modulo 125 and 8.

Let $\varphi(m)$ count the integers from 1 through m that are relatively prime to m. Euler's totient theorem states that, whenever a and m are relatively prime,

$$a^{\varphi(m)} \equiv 1 \pmod{m}.$$

Since $125 = 5^3$, the numbers not relatively prime to it are exactly the multiples of 5. There are $\frac{125}{5} = 25$ of these from 1 through 125. Therefore

$$\varphi(125) = 125\left(1 - \frac{1}{5}\right) = 100.$$

The numbers 2 and 125 are relatively prime, so the theorem gives $2^{100} \equiv 1 \pmod{125}$. Squaring gives $2^{200} \equiv 1 \pmod{125}$, and subtracting 1 gives $N \equiv 0 \pmod{125}$. Also, 2^{200} is divisible by 8, so $N \equiv 7 \pmod{8}$.

Among the multiples of 125 from 0 through 999, only 375 is 7 modulo 8. Therefore the last three digits are 375.

Thus, the correct answer is **C**.

24. Let $\triangle ABC$ be a scalene triangle with incenter I. (Recall that the incenter is the intersection of the three angle bisectors.) The distances from I to the midpoints of the sides of $\triangle ABC$ are $\sqrt{11}$, $\sqrt{14}$, and $\sqrt{19}$. What is the sum of the possible areas of the circle inscribed in $\triangle ABC$?

A 10π

B $\frac{44\pi}{3}$

C $\frac{58\pi}{3}$

D $\frac{88\pi}{3}$

E 30π

Small hint:

The two tangent segments from a vertex to the incircle have equal lengths

Big hint:

The three distances from side midpoints to tangency points are half the pairwise differences of those lengths

Solution:

Let the tangent lengths from the three vertices be x, y, and z, and put $t = r^2$, where r is the inradius. On the side between vertices with tangent lengths x and y, the side midpoint is distance $\frac{|x-y|}{2}$ from the tangency point. Its squared distance from I is therefore $t + \frac{(x-y)^2}{4}$.

Place x, y, and z on a number line. Of their three pairwise distances, the largest equals the sum of the other two. Hence

$$\sqrt{19-t} = \sqrt{14-t} + \sqrt{11-t}.$$

Squaring twice yields

$$\begin{aligned} t-6 &= 2\sqrt{(14-t)(11-t)} \\ (t-6)^2 &= 4(14-t)(11-t) \\ 0 &= (t-10)(3t-58) \end{aligned}$$

Since $0 < t \leq 11$, we get $t = 10$. The incircle area is 10π.

Thus, the correct answer is **A**.

25. A *finite integer tower* is an expression of the form

$$a_1^{a_2^{a_3^{\cdot^{\cdot^{\cdot^{a_n}}}}}}$$

where $n \geq 1$ and $a_1, a_2, \ldots, a_n$ are positive integers greater than 1. (Recall that exponentiation is performed from the top down. For example, 3^{2^3} equals 3^8, not 9^3.) Two towers are considered different if they have a different number of layers (values of n) or at least one pair of numbers in corresponding positions differ.

Define $f(x)$ to be the number of different finite integer towers whose value is x. (For example, $f(16) = 4$ because 4 towers evaluate to 16, namely $16, 2^4, 4^2$, and 2^{2^2}.) What is $f(2^{8192})$?

A 17

B 32

C 42

D 43

E 47

Small hint:

In a multilayer tower equal to 2^m, the bottom entry must be a power of 2

Big hint:

If the upper tower evaluates to d, then $d \mid m$ and the bottom entry is $2^{\frac{m}{d}}$

Solution:

Write $F(m) = f(2^m)$. There is one single-entry tower. In any longer tower, the bottom entry must be a power of 2, since it has no other prime factors. Write it as 2^e. If the upper part has value $d \geq 2$, then $ed = m$. Thus $d \mid m$, and the bottom entry is uniquely $2^{\frac{m}{d}}$. Consequently

$$F(m) = 1 + \sum_{\substack{d \mid m \\ d \geq 2}} f(d).$$

Since $8192 = 2^{13}$, this gives

$$F(8192) = 1 + \sum_{j=1}^{13} F(j).$$

For $2 \leq d \leq 13$, the only perfect powers are $4, 8$, and 9, each having two towers; all other such d have one. For example,

$$\begin{aligned} F(8) &= 1 + f(2) + f(4) + f(8) \\ &= 1 + 1 + 2 + 2 = 6. \end{aligned}$$

Applying the same divisor formula gives

j	1	2	3	4	5	6	7
$F(j)$	1	2	2	4	2	4	2

j	8	9	10	11	12	13
$F(j)$	6	4	4	2	7	2

These values sum to 42, so $f(2^{8192}) = 43$.

Thus, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc12/2026B