2026 AMC 12A Solutions

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1. A sign in a parking lot states that the total price includes a 27% tax. To the nearest whole number, what percentage of the amount a customer pays is this tax?

A 21%

B 22%

C 27%

D 37%

E 73%

Small hint:

A $\$100$ price means $\$27$ tax and $\$127$ paid

Big hint:

$\frac{27}{127}$ is close to $\frac{25}{125}$

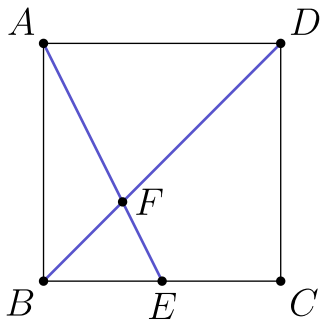
Solution:

Say the price before tax is $\$100$. Then the tax is $\$27$, and the customer pays $\$127$ in all. So the tax is $\frac{27}{127}$ of the amount paid.

This is close to $\frac{25}{125} = \frac{1}{5}$, so the tax is near 20%. Now $0.21 \times 127 = 26.67$ and $0.22 \times 127 = 27.94$, and 27 is closer to 26.67. So to the nearest whole number the tax is 21% of the amount paid.

Thus, the correct answer is **A**.

2. Square $ABCD$ has side length 24. Let E be the midpoint of $\overline{BC}$, and let F be the intersection of $\overline{AE}$ and $\overline{BD}$.



What is the area of $\triangle BEF$?

A 36

B 40

C 45

D 48

E 52

Small hint:

$\overline{BE} \parallel \overline{AD}$, so triangles FBE and FDA are similar

Big hint:

BE is half of AD, so $\triangle FBE$ is half as tall as $\triangle FDA$

Solution:

Since $\overline{BE} \parallel \overline{AD}$, triangles FBE and FDA are similar, and BE is half of AD. So the height of $\triangle FBE$ is half the height of $\triangle FDA$. The two heights add up to the side 24, so F is one third of the way up, at height 8.

Then $\triangle BEF$ has area $\frac{1}{2} \times 12 \times 8 = 48$.

Alternatively, $\overline{AE}$ is a median of $\triangle ABC$. The diagonal $\overline{BD}$ passes through the center of the square, which is the midpoint of $\overline{AC}$, so it contains the median from B. So F is

the centroid of $\triangle ABC$. The three medians cut $\triangle ABC$ into six triangles of equal area, and $\triangle BEF$ is one of them. So its area is $\frac{1}{6}$ of $\frac{1}{2} \times 24 \times 24 = 288$, which is 48.

Thus, the correct answer is **D**.

3. Half of the internal angles of a polygon measure 150° each, and the other half of its internal angles measure 190° each. How many sides does the polygon have?

A 36

B 40

C 60

D 72

E 180

Small hint:

Exterior angles, each 180° minus the interior angle, sum to 360° even when some are negative

Big hint:

The interior angles average 170°

Solution:

The exterior angle at each corner is 180° minus the interior angle: 30° at the 150° angles and -10° at the 190° angles, where the boundary turns back the other way. Going once around any polygon turns through 360° in all, so the exterior angles add up to 360°.

The interior angles average 170°, so the exterior angles average 10°. With n sides, $10n = 360$, so $n = 36$.

Thus, the correct answer is **A**.

4. Let $a_1, a_2, a_3, \ldots$ be a sequence with $a_1 = 94, a_2 = 190$, and a_n equal to the arithmetic mean of the previous $n-1$ terms for $n \geq 3$. What is the value of a_{10}?

A 47

B 95

C 142

D 284

E 564

Small hint:

a_3 is the mean of 94 and 190

Big hint:

Appending the current mean leaves the mean unchanged

Solution:

First, $a_3 = \frac{94+190}{2} = 142$. Now the first three terms have mean 142, because a_3 equals the mean of the two terms before it. So $a_4 = 142$.

Each time we append a term equal to the current mean, the mean stays the same. Therefore every term from a_3 on equals 142, and $a_{10} = 142$.

Thus, the correct answer is **C**.

5. How many ordered triples of positive integers (x, y, z) satisfy both of the following equations?

$$x + y + z = 31$$
$$x + 3y + 3z = 41$$

A 4

B 5

C 6

D 7

E 8

Small hint:

Subtract the first equation from the second

Big hint:

That fixes $y + z$ and x. Count the ways to split $y + z$ into two positive integers.

Solution:

Subtracting the first equation from the second gives $2y + 2z = 10$, so $y + z = 5$ and $x = 31 - 5 = 26$. The pair (y, z) can be $(1, 4), (2, 3), (3, 2)$, or $(4, 1)$, so there are 4 triples.

Thus, the correct answer is **A**.

6. For what odd positive integer n is the mean of $1, 4, 9, 16, \ldots, n^2$ exactly 32% greater than its median?

A 19

B 39

C 49

D 89

E 169

Small hint:

The mean is $\frac{(n+1)(2n+1)}{6}$ and the median is $\left(\frac{n+1}{2}\right)^2$

Big hint:

Their ratio is $\frac{2(2n+1)}{3(n+1)}$. Set it to $\frac{33}{25}$.

Solution:

The mean of the n squares is $\frac{(n+1)(2n+1)}{6}$, and since n is odd, the median is the middle square, $\left(\frac{n+1}{2}\right)^2$. The factor $n+1$ cancels in their ratio:

$$\frac{(n+1)(2n+1)}{6} \cdot \frac{4}{(n+1)^2} = \frac{2(2n+1)}{3(n+1)}.$$

Setting this equal to $\frac{33}{25}$ gives $50(2n+1) = 99(n+1)$, so $n = 49$.

Thus, the correct answer is **C**.

7. What is the value of $(1+i)^{20}+(1-i)^{20}$, where $i=\sqrt{-1}$?

A -2048

B -1024

C 0

D 1024

E 2048

Small hint:

$(1+i)^2=2i$ and $(1-i)^2=-2i$

Big hint:

Square again to get $(1\pm i)^4$, then take the 5th power

Solution:

Since $(1+i)^2=2i$, squaring again gives $(1+i)^4=(2i)^2=-4$. Likewise $(1-i)^2=-2i$ and $(1-i)^4=(-2i)^2=-4$. So each term is $(-4)^5=-1024$, and the sum is -2048.

Thus, the correct answer is **A**.

8. At Euclid High School, 25 students registered to take the AMC 10 and 75 students registered to take the AMC 12. (No student registered to take both.) However, not all 100 students showed up on test day. To compute the percentage of those students who actually took one of the exams, Euclid High School performed the following incorrect two-step process:

Step 1: The school computed both the percentage of AMC 10 registrants who actually took that exam and the percentage of AMC 12 registrants who actually took that exam.

Step 2: The school took the average of the previous two numbers to obtain a final percentage.

Suppose that Euclid High School's process yielded an answer of 60%. What is the minimum number of students who might have actually taken one of the exams?

A 30

B 35

C 40

D 50

E 60

Small hint:

With x AMC 10 and y AMC 12 takers, $\frac{1}{2}\left(\frac{x}{25}+\frac{y}{75}\right)=\frac{60}{100}$

Big hint:

So $3x+y=90$. Minimize $x+y$.

Solution:

Say x of the AMC 10 registrants and y of the AMC 12 registrants took the exams. The school's process gives $\frac{1}{2}\left(\frac{x}{25}+\frac{y}{75}\right)=\frac{3}{5}$. Multiplying by 150 gives $3x+y=90$.

Then $x+y=90-2x$, which is smallest when x is as large as possible. Since $x\leq 25$, take $x=25$ and $y=15$. So the minimum is $25+15=40$.

Thus, the correct answer is **C**.

9. How many ordered pairs (x, y) of real numbers satisfy the following system of equations?

$$\begin{aligned} \left|x^2 - y^2\right| &= 4 \\ \frac{16x}{y} &= xy \end{aligned}$$

A 2

B 4

C 6

D 8

E 10

Small hint:

Multiply the second equation by $y \neq 0$: $x(y^2 - 16) = 0$

Big hint:

So $x = 0$ or $y = \pm 4$. Substitute each into the first equation.

Solution:

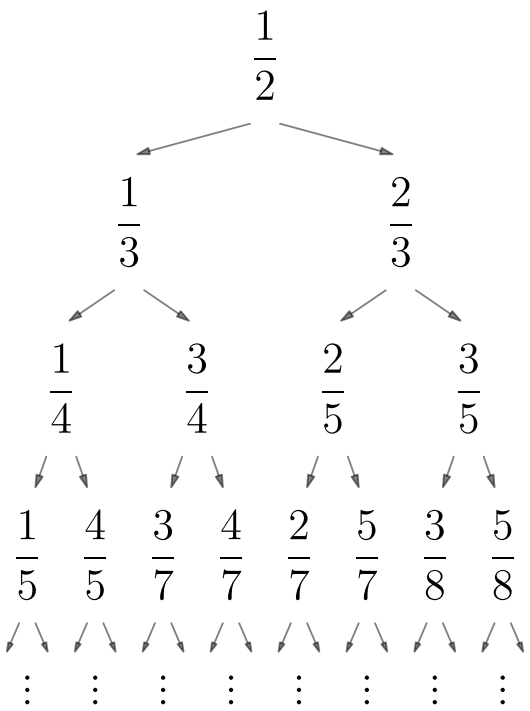
Here $y \neq 0$. Multiplying the second equation by y gives $16x = xy^2$, or $x(y^2 - 16) = 0$. So $x = 0$ or $y = \pm 4$.

If $x = 0$, the first equation says $y^2 = 4$, so $y = \pm 2$. That gives 2 pairs.

If $y = \pm 4$, the first equation says $\left|x^2 - 16\right| = 4$, so $x^2 = 20$ or $x^2 = 12$. That gives 4 values of x for each of the 2 values of y, or 8 pairs. None of them has $x = 0$, so the two cases do not overlap, and there are $2 + 8 = 10$ pairs.

Thus, the correct answer is **E**.

10. The tree diagram below has nodes labeled with rational numbers. The top node, at level 0, is $\frac{1}{2}$. Each node $\frac{a}{b}$ sprouts two nodes in the level below it, $\frac{a}{a+b}$ and $\frac{b}{a+b}$. For example, level 1 has nodes $\frac{1}{1+2} = \frac{1}{3}$ and $\frac{2}{1+2} = \frac{2}{3}$.



At what level does the node $\frac{20}{27}$ appear in the diagram?

A 5

B 8

C 12

D 26

E The node $\frac{20}{27}$ does not appear in the diagram.

Small hint:

Going up a level, $\frac{p}{q}$ becomes a fraction made of p and $q - p$

Big hint:

Starting from 20 and 27, keep subtracting the smaller number from the larger until you reach 1 and 2

Solution:

Every node is a fraction less than 1 in lowest terms. If a node is $\frac{p}{q}$, its parent is made of p and $q-p$, in whichever order gives a fraction less than 1. So each step up replaces the pair of numbers by the smaller one and their difference, as in the Euclidean algorithm.

Starting from $(20, 27)$, the pairs are $(7, 20), (7, 13), (6, 7), (1, 6), (1, 5), (1, 4), (1, 3)$, and $(1, 2)$. That is 8 steps up to $\frac{1}{2}$, so $\frac{20}{27}$ is at level 8.

Thus, the correct answer is **B**.

11. An urn contains 6 red balls, 4 black balls, and 4 white balls. Suppose 7 balls are drawn at random without replacement and a score is computed as follows:

- 1 point is given for each red ball up to a maximum of 3 points.
- 1 point is given for each black ball up to a maximum of 2 points.
- 1 point is given for each white ball up to a maximum of 2 points.

For example, drawing 1 red ball, 4 black balls, and 2 white balls would earn a score of $1 + 2 + 2 = 5$. What is the probability of getting the lowest possible score?

A $\frac{1}{858}$

B $\frac{5}{3432}$

C $\frac{1}{312}$

D $\frac{2}{429}$

E $\frac{5}{264}$

Small hint:

The score is 7 minus the balls beyond their color's cap. Waste as many balls as possible.

Big hint:

At most 3 balls can be wasted: 6 reds and 1 other, or 4 of one other color and 3 of the last

Solution:

Call a ball wasted if it is beyond its color's cap. Then the score is 7 minus the number of wasted balls, so we want to waste as many as possible.

Wasting 4 is impossible. With 6 reds, 3 reds are wasted and the 1 other ball is not. With 5 reds, 2 reds are wasted and the 2 other balls waste nothing. With 4 reds, 1 red is

wasted and the 3 other balls waste at most 1. With at most 3 reds, only black and white can be wasted, and wasting 2 of each would take 8 balls. So at most 3 are wasted, and the lowest score is 4.

Exactly 3 are wasted with 6 reds and 1 of the 8 other balls (8 ways), with 4 black and 3 white (4 ways), or with 3 black and 4 white (4 ways). That is 16 of the $\binom{14}{7} = 3432$ equally likely draws, for a probability of $\frac{16}{3432} = \frac{2}{429}$.

Thus, the correct answer is **D**.

12. Ethan writes down five congruences:

$$\begin{aligned} x &\equiv 0 \pmod 2, \\ x &\equiv 0 \pmod 3, \\ x &\equiv 1 \pmod 4, \\ x &\equiv 5 \pmod 6, \\ x &\equiv 7 \pmod{12}. \end{aligned}$$

How many positive integers less than or equal to 2026 satisfy at least one of these five congruences?

A 1688

B 1856

C 1857

D 2025

E 2026

Small hint:

Every modulus divides 12, so only the remainder of x mod 12 matters

Big hint:

Evens and multiples of 3 are covered. Check the remainders $1, 5, 7$, and 11.

Solution:

Each modulus divides 12, so whether x satisfies a congruence depends only on the remainder of x when divided by 12. The first congruence covers the remainders $0, 2, 4, 6, 8$, and 10. The second adds 3 and 9. The third adds 1 and 5, the fourth adds 11, and the fifth adds 7.

All twelve remainders are covered, so every positive integer satisfies at least one congruence. The answer is 2026.

Thus, the correct answer is **E**.

13. A board has 4 spaces in a row, which can be filled with numbers. The sides of a fair, 8-sided die are numbered 1 through 8. David rolls the die 4 times. After each roll, he randomly chooses an empty spot to write the number shown on the die. What is the probability that the numbers on the board form a strictly increasing sequence when read from left to right?

A $\frac{1}{128}$

B $\frac{5}{432}$

C $\frac{35}{2048}$

D $\frac{35}{1024}$

E $\frac{1}{24}$

Small hint:

The rolls must be distinct, and then one placement is increasing

Big hint:

The placement is one of 24 equally likely orders, whatever was rolled

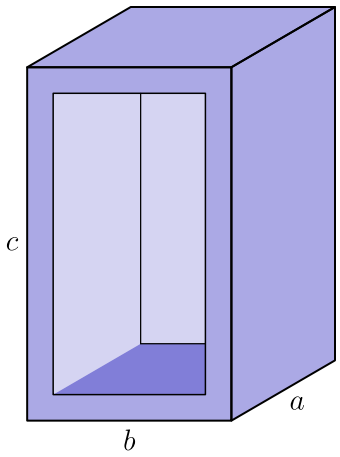
Solution:

David's choices of spots put the four rolls on the board in one of $4! = 24$ orders, each equally likely and unrelated to the numbers rolled.

A strictly increasing board needs four different numbers. That happens with probability $\frac{8\times7\times6\times5}{8^4} = \frac{105}{256}$. Then exactly one of the 24 orders is increasing. The probability is $\frac{105}{256} \times \frac{1}{24} = \frac{35}{2048}$.

Thus, the correct answer is **C**.

14. Three identical boxes, each with one open face, are made from 1-inch-thick wood. The outer dimensions of each box are a, b, and c inches, where $a < b < c$, and each open face appears on a face with outer dimensions $b \times c$.



It happens that if the open faces of two of the boxes are held together, then the space inside can exactly contain the (rotated) third box, with no room to spare. What is $a + b + c$?

A 20

B 22

C 24

D 26

E 28

Small hint:

One box's inside is $(a-1) \times (b-2) \times (c-2)$

Big hint:

Two boxes make $(2a-2) \times (b-2) \times (c-2)$. Match this to a, b, and c in some order.

Solution:

Each box has walls 1 inch thick on every face except the open $b \times c$ face, so its inside measures $(a-1) \times (b-2) \times (c-2)$. Holding two boxes open face to open face makes a space of $(2a-2) \times (b-2) \times (c-2)$.

This must be a, b, and c in some order. Since $b-2$ and $c-2$ are both less than c, the number c must be $2a-2$. Then $b-2$ and $c-2$ are a and b in some order, and $b-2 \neq b$, so $b-2=a$ and $c-2=b$.

So $b=a+2$ and $c=a+4=2a-2$, which gives $a=6, b=8$, and $c=10$. Then $a+b+c=24$.

Thus, the correct answer is **C**.

15. Suppose that $\sin a = \frac{1}{4}$ and $\sin b = \frac{1}{8}$ with $0 \leq b < a \leq \frac{\pi}{2}$. What is the greatest integer less than $\log_2\big(\sin(a+b) \cdot \sin(a-b)\big)$?

A -5

B -4

C 0

D 1

E 5

Small hint:

$2 \sin u \sin v = \cos(u-v) - \cos(u+v)$

Big hint:

The product is $\sin^2 a - \sin^2 b$. See which powers of $\frac{1}{2}$ it lies between.

Solution:

By the product-to-sum identity $2 \sin u \sin v = \cos(u-v) - \cos(u+v)$ and the double-angle identity $\cos(2\theta) = 1 - 2\sin^2\theta$,

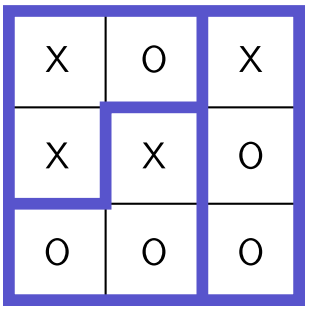
$$\begin{aligned} &\sin(a+b)\sin(a-b) \\ &= \frac{\cos(2b) - \cos(2a)}{2} \\ &= \sin^2 a - \sin^2 b. \end{aligned}$$

This is $\frac{1}{16} - \frac{1}{64} = \frac{3}{64}$. Since $\frac{1}{32} < \frac{3}{64} < \frac{1}{16}$, its base-$2$ logarithm is between -5 and -4. The greatest integer less than it is -5.

Thus, the correct answer is **A**.

16. Gerry is playing the Triples Game. Her task is to partition the 3×3 grid of squares shown at the top into three sections, each containing three adjoining squares. (Three squares are adjoining if each shares an edge with at least one of the other two.) One point will be awarded for each section containing more X's than O's. Shown below it is a sample partition that will earn one point in total. A partition that earns two or more points will win the game. What fraction of the possible partitions of this particular grid will result in a win for Gerry?

X	O	X
X	X	O
O	O	O



A $\frac{3}{10}$

B $\frac{1}{3}$

C $\frac{2}{5}$

D $\frac{4}{9}$

E $\frac{1}{2}$

Small hint:

Only 4 squares hold X's. A win needs two sections with 2 X's each, so the third section is all O's.

Big hint:

Only the bottom row and one L-shape consist of three O's. Check the partitions that contain each.

Solution:

Each section is a straight line of three squares or an L-shape. An L-shape fits in a 2×2 block, so it covers at most one corner of the grid, and a middle row or column covers none. The three sections cover all four corners, so one of them is a full row or column along the edge.

That leaves a 2×3 rectangle, which splits in 3 ways: into two straight pieces, or into two L-shapes in 2 ways. Four edges give $4 \times 3 = 12,$ but the three-rows and three-columns partitions are each counted twice, so there are 10 partitions.

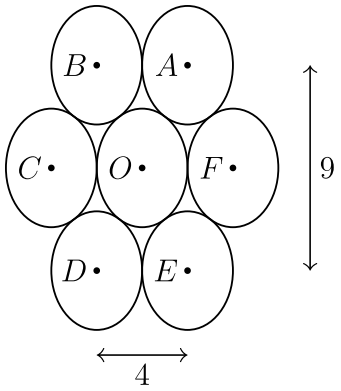
The grid has only 4 X's, and a section earns a point only with at least 2 of them. So a winning partition has two sections with 2 X's each and a third section of three O's. Only two sections consist of three O's: the bottom row, and the L made of the middle-right, bottom-center, and bottom-right squares.

With the bottom row, the top two rows split as two rows (XOX and XXO) or as two L-shapes (XOX and XXO again), both wins, or as XXX and OXO, which earns only 1 point. With the L, the other six squares split only as the top row plus an L (XOX and XXO) or as the left column plus an L (XXO and OXX), both wins.

So 4 of the 10 partitions win, a fraction of $\frac{2}{5}$.

Thus, the correct answer is **C**.

17. An engineer is designing a closely packed field of 7 wind turbines for electricity production. Each turbine, whose center is marked by a labeled dot in the figure, occupies one of 7 congruent elliptical regions. Turbines in the same row have centers that are 4 units apart, and turbines in the same column have centers that are 9 units apart, as shown in the figure below.



The major axis of each ellipse is $m\sqrt{p}$, where m and p are positive integers and p is not divisible by the square of any prime. What is $m + p$?

A 6

B 8

C 9

D 10

E 12

Small hint:

Each ellipse is 4 wide. Squeeze vertically to make them circles of radius 2.

Big hint:

O and A are 2 apart across and 4.5 apart vertically, and their ellipses touch

Solution:

Turbines in the same row touch halfway between their centers, so each ellipse has horizontal semi-axis 2. Call the vertical semi-axis s.

Squeeze the whole picture vertically by the factor $\frac{2}{s}$. Every ellipse becomes a circle of radius 2, and ellipses that touched still touch. The centers O and A were 2 apart across and 4.5 apart up and down, so now they are 2 across and $\frac{9}{s}$ up and down. Touching circles of radius 2 have centers 4 apart, so

$$4 + \frac{81}{s^2} = 16,$$

which gives $s^2 = \frac{27}{4}$ and $s = \frac{3\sqrt{3}}{2}$.

Since $s > 2$, the major axis is vertical, with length $2s = 3\sqrt{3}$. So $m + p = 3 + 3 = 6$.

Thus, the correct answer is **A**.

18. A sphere with center O and radius 9 cuts three mutually perpendicular planes that meet at point P, forming three circular cross-sections on these planes, which have radii $7, 4,$ and 3. What is OP?

A 10

B 11

C 12

D 13

E 14

Small hint:

A plane at distance d from O cuts a circle of radius $\sqrt{81 - d^2}$

Big hint:

With P at the origin and the planes as coordinate planes, these distances are the coordinates of O

Solution:

A plane at distance d from O meets the sphere in a circle of radius $\sqrt{81 - d^2}$. So the squared distances from O to the three planes are $81 - 49 = 32, 81 - 16 = 65,$ and $81 - 9 = 72$.

Put P at the origin with the planes as the coordinate planes. Then these distances are the coordinates of O (up to sign), so $OP^2 = 32 + 65 + 72 = 169$ and $OP = 13$.

Thus, the correct answer is **D**.

19. How many values of x between 0 and 2π, inclusive, satisfy the equation

$$\sin x + \sin(2x) + \sin(3x) = 1 + \cos x + \cos(2x)?$$

A 1

B 2

C 4

D 6

E 8

Small hint:

$\sin x + \sin(3x) = 2\sin(2x)\cos x$ and $1 + \cos(2x) = 2\cos^2 x$

Big hint:

Both sides then share the factor $2\cos x + 1$

Solution:

Since $\sin x + \sin(3x) = 2\sin(2x)\cos x$, the left side is $\sin(2x)(2\cos x + 1)$. Since $1 + \cos(2x) = 2\cos^2 x$, the right side is $\cos x\,(2\cos x + 1)$. So the equation becomes $(2\cos x + 1)(\sin(2x) - \cos x) = 0$, and since $\sin(2x) = 2\sin x \cos x$,

$$(2\cos x + 1)\cos x\,(2\sin x - 1) = 0.$$

In $[0, 2\pi]$, $\cos x = -\frac{1}{2}$ gives $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, $\cos x = 0$ gives $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, and $\sin x = \frac{1}{2}$ gives $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. These 6 values are all different.

Thus, the correct answer is **D**.

20. How many polynomials $f(x)$ of degree 3 satisfying

$$f\left(3+\sqrt{5}\right)=f\left(3-\sqrt{5}\right)$$

are there that have leading coefficient equal to 1 and all of whose coefficients are integers between -10 and 10, inclusive?

A 21

B 42

C 63

D 84

E 105

Small hint:

Put $x=3+t$, so that the two inputs are $t=\pm\sqrt{5}$

Big hint:

Replacing t by $-t$ changes only the odd powers of t

Solution:

Write $f(x)=x^3+ax^2+bx+c$. The inputs $3\pm\sqrt{5}$ are symmetric about 3, so put $x=3+t$. Expanding $(3+t)^3$, $a(3+t)^2$, and $b(3+t)$, the coefficient of t^3 is 1 and the coefficient of t is $27+6a+b$.

Replacing t by $-t$ flips the signs of the odd powers of t and leaves the even powers alone, so

$$\begin{aligned} &f(3+t)-f(3-t)\\ &=2t^3+2(27+6a+b)t. \end{aligned}$$

At $t=\sqrt{5}$ this is $2\sqrt{5}\,(5+27+6a+b)$, which must be 0. So $b=-32-6a$.

For $-10\le b\le 10$ we need $-7\le a\le -4$, giving $b=10,4,-2$, and -8, all in range. The constant c can be any of the 21 integers from -10 to 10. So there are $4\times 21=$

84 polynomials.

Thus, the correct answer is **D**.

21. Six dogs find a large cookie in the shape of a regular hexagon $ABCDEF$. To divide the cookie, the dogs agree to choose a point P uniformly at random in the interior of the hexagon. The dogs will then cut out the segments $\overline{PA}, \overline{PB}, \overline{PC}, \overline{PD}, \overline{PE}$, and $\overline{PF}$, dividing the cookie into six triangles, one for each dog. A dog will be happy if and only if its piece has at least $\frac{1}{10}$ of the total area of the hexagon. What is the probability that all six dogs will be happy?

A $\frac{\sqrt{3}}{50}$

B $\frac{1}{25}$

C $\frac{3\sqrt{3}}{50}$

D $\frac{4}{25}$

E $\frac{2}{5}$

Small hint:

With side s and center-to-side distance h, the area is $3sh$. If P is t from a side, that side's triangle has area $\frac{1}{2}st$.

Big hint:

Each dog needs P at least $\frac{3}{5}h$ from its side. That region is a smaller hexagon with the same center.

Solution:

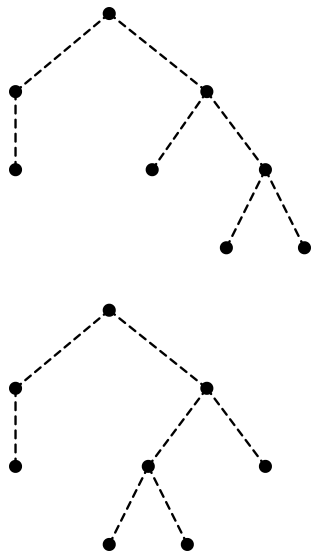
Let the hexagon have side s and let h be the distance from its center to each side. The hexagon splits into six triangles from the center, so its area is $6 \cdot \frac{1}{2}sh = 3sh$. The triangle on a side whose distance from P is t has area $\frac{1}{2}st$.

That dog is happy when $\frac{1}{2}st \geq \frac{3sh}{10}$, or $t \geq \frac{3}{5}h$. So all six dogs are happy when P is at least $\frac{3}{5}h$ from every side, which means P lies in a regular hexagon with the same center whose sides are $h - \frac{3}{5}h = \frac{2}{5}h$ from the center.

That hexagon is the original scaled by $\frac{2}{5}$, so the probability is $\left(\frac{2}{5}\right)^2 = \frac{4}{25}$.

Thus, the correct answer is **D**.

22. A flock of 8 birds is flying in formation, led by a single leader bird, which has exactly 2 followers immediately behind. Each non-leader bird follows exactly one bird, and has $0, 1,$ or 2 followers immediately behind. Two possible formations are shown in the figure below. (When a bird has two followers, swapping its left and right subflocks can produce a different formation, as shown in the figure.)



What is the total number of possible formations?

A 48

B 52

C 76

D 132

E 196

Small hint:

Let $T(n)$ count subflocks of n birds. With 2 followers, the other $n-1$ birds split into an ordered left and right pair.

Big hint:

The leader's subflocks have i and $7-i$ birds. Compute $\sum_{i=1}^{6} T(i)\,T(7-i)$.

Solution:

Let $T(n)$ be the number of ways to arrange a subflock of n birds, each with $0, 1,$ or 2 followers. A single follower has no left or right, while two followers form an ordered left and right pair. So $T(1) = 1$ and

$$T(n) = T(n-1) + \sum_{i=1}^{n-2} T(i)\,T(n-1-i).$$

This gives $T(2) = 1, T(3) = 2, T(4) = 4, T(5) = 4 + 5 = 9,$ and $T(6) = 9 + 12 = 21.$

The leader has exactly two followers, with subflocks of i and $7 - i$ birds. The number of formations is

$$\begin{aligned} &T(1)T(6) + T(2)T(5) \\ &+ T(3)T(4) + T(4)T(3) \\ &+ T(5)T(2) + T(6)T(1) \\ &= 21 + 9 + 8 + 8 + 9 + 21 \\ &= 76. \end{aligned}$$

Thus, the correct answer is **C**.

23. There is a unique real number x between 0 and 1 such that $x^3 + x^2 = 1$. For that x, there exist unique integers a, b, and c such that the infinite product

$$(x+1)(x^2+1)(x^4+1)$$
$$\cdot (x^8+1)(x^{16}+1)\cdots$$

equals $ax^2 + bx + c$. What is $a + b + c$?

A 2

B 3

C 4

D 5

E 6

Small hint:

Multiply by $1 - x$. The factors collapse, starting with $(1-x)(1+x) = 1 - x^2$.

Big hint:

So the product is $\frac{1}{1-x}$. Use $1 - x^3 = x^2$ and $x^2(x+1) = 1$.

Solution:

Multiplying the first n factors by $1 - x$ collapses them one at a time:

$$(1-x)(1+x)(1+x^2)$$
$$\cdots\left(1+x^{2^{n-1}}\right) = 1 - x^{2^n}.$$

Since $0 < x < 1$, the right side tends to 1, so the infinite product equals $\frac{1}{1-x}$.

To turn $\frac{1}{1-x}$ into a quadratic, use $x^3 + x^2 = 1$ twice. First, $1 - x^3 = x^2$, and $1 - x^3 = (1-x)(1+x+x^2)$, so $\frac{1}{1-x} = \frac{1+x+x^2}{x^2}$. Second, $x^2(x+1) = 1$, so $\frac{1}{x^2} = x + 1$. Therefore

$$\begin{aligned}\frac{1}{1-x} &= (1+x+x^2)(1+x) \\ &= 1+2x+2x^2+x^3 \\ &= x^2+2x+2,\end{aligned}$$

using $x^3 = 1 - x^2$ in the last step.

So $a = 1, b = 2,$ and $c = 2,$ and $a + b + c = 5.$

Thus, the correct answer is **D**.

24. Trapezoid $ABCD$ has $AB = 4, CD = 5$, and $\angle DAB = \angle ABC = 90^\circ$. Circles ω_1 and ω_2 lie inside the trapezoid as follows: ω_1 is tangent to $\overline{AB}$, $\overline{AD}$, and $\overline{DC}$; ω_2 is tangent to $\overline{AB}$, $\overline{BC}$, and $\overline{DC}$; and ω_1 and ω_2 are externally tangent to each other. The distance between the centers of ω_1 and ω_2 is $m - n\sqrt{p}$, where m, n, and p are positive integers such that p is not divisible by the square of any prime. What is $m + n + p$?

A 30

B 47

C 53

D 62

E 74

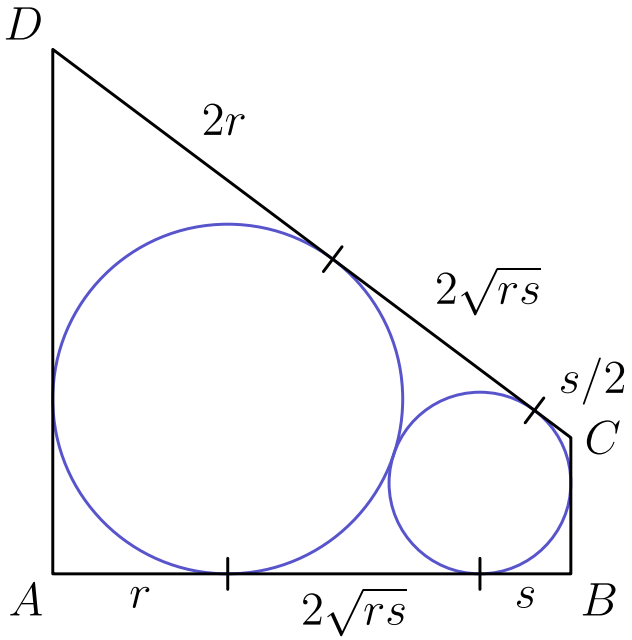
Small hint:

The perpendicular from C to $\overline{AD}$ cuts off a 3-4-5 triangle. Find the tangent lengths from D and C.

Big hint:

On both $\overline{AB}$ and $\overline{DC}$, the two touch points are $2\sqrt{rs}$ apart

Solution:



Since $\angle A$ and $\angle B$ are right angles, $\overline{AD} \parallel \overline{BC}$. Say $\overline{AD}$ is the longer base. The perpendicular from C to $\overline{AD}$ has length 4, so it cuts off a right triangle with legs 3 and 4 and hypotenuse $\overline{DC}$. Its incircle has radius $\frac{3+4-5}{2} = 1$ and touches the two sides through D at points $\frac{3+5-4}{2} = 2$ from that vertex. So $\tan\frac{D}{2} = \frac{1}{2}$, and since the angles at C and D add up to 180°, we get $\tan\frac{C}{2} = 2$.

A circle of radius ρ inside an angle θ touches both sides at distance $\rho/\tan\frac{\theta}{2}$ from the vertex. So if ω_1 and ω_2 have radii r and s, the tangent length from D to ω_1 is $2r$, and from C to ω_2 it is $\frac{s}{2}$. The tangent lengths from A and B are r and s.

Both $\overline{AB}$ and $\overline{DC}$ are common external tangents of the two touching circles. On each of them the touch points are $2\sqrt{rs}$ apart, since $(r+s)^2 - (r-s)^2 = 4rs$. Adding up the pieces of each side, as in the figure,

$$r + 2\sqrt{rs} + s = 4,$$
$$2r + 2\sqrt{rs} + \frac{s}{2} = 5.$$

Subtracting the first equation from the second gives $r - \frac{s}{2} = 1$, so $s = 2r - 2$. The first equation says $(\sqrt{r} + \sqrt{s})^2 = 4$, so $\sqrt{r} + \sqrt{s} = 2$.

Then $\sqrt{2r-2} = 2 - \sqrt{r}$. Squaring gives $2r - 2 = 4 - 4\sqrt{r} + r$, or $r + 4\sqrt{r} - 6 = 0$, so $\sqrt{r} = \sqrt{10} - 2$. Thus $r = 14 - 4\sqrt{10}$ and $s = 2r - 2 = 26 - 8\sqrt{10}$.

The distance between the centers is $r + s = 40 - 12\sqrt{10}$, so $m + n + p = 40 + 12 + 10 = 62$.

Thus, the correct answer is **D**.

25. Let m be a positive integer such that there are at least 50 positive integers x less than m for which $x^2 - 1$ is a multiple of m. What is the sum of the digits of the least possible value of m?

A 12

B 15

C 16

D 18

E 22

Small hint:

Mod an odd prime power, $x^2 \equiv 1$ has 2 solutions. Mod $2, 4$, and 2^k with $k \geq 3$, it has 1, 2, and 4.

Big hint:

By the Chinese Remainder Theorem the counts multiply, so the count must reach 64

Solution:

We are counting the solutions of $x^2 \equiv 1 \pmod{m}$ among $1, 2, \ldots, m-1$. For $m > 1$ this is every solution modulo m, because $x = 0$ is never one.

Modulo an odd prime power p^k, we need p^k to divide $(x-1)(x+1)$. The prime p cannot divide both factors, since they differ by 2, so $x \equiv \pm 1$: there are 2 solutions. Modulo 2 there is 1 solution, modulo 4 there are 2, and modulo 2^k with $k \geq 3$ there are 4. There x is odd, and of the even numbers $x-1$ and $x+1$ one is not a multiple of 4, so the other is a multiple of 2^{k-1}. That gives $x \equiv \pm 1 \pmod{2^{k-1}}$, which is ± 1 and $2^{k-1} \pm 1$ modulo 2^k.

By the Chinese Remainder Theorem, the number of solutions modulo m is the product over the prime powers in m. So it is a power of 2, and to be at least 50 it must be at least $64 = 2^6$.

With a factor 2^k for $k \geq 3$ we need at least 4 odd primes, and the smallest choice is $8 \times 3 \times 5 \times 7 \times 11 = 9240$. With exactly 4 dividing m we need 5 odd primes, so

$m \geq 4 \times 3 \times 5 \times 7 \times 11 \times 13 = 60060$. With at most one factor of 2 we need 6 odd primes, so $m \geq 3 \times 5 \times 7 \times 11 \times 13 \times 17 = 255255$.

The least m is 9240, and its digit sum is $9 + 2 + 4 + 0 = 15$.

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc12/2026A