2024 AMC 12B Solutions

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1. In a long line of people arranged left to right, the 1013th person from the left is also the 1010th person from the right. How many people are in the line?

- A 2021
- B 2022
- C 2023
- D 2024
- E 2025

Solution:

The target person has $1013 - 1 = 1012$ people to the left and $1010 - 1 = 1009$ people to the right. Including the person themself, the line has $1012 + 1009 + 1 = 2022$ people.

Thus, the correct answer is **B**.

2. What is $10! - 7! \cdot 6!$?

A -120

B 0

C 120

D 600

E 720

Solution:

Note that $6! = 720 = 8 \cdot 9 \cdot 10$. Therefore

$$7! \cdot 6! = 7! \cdot (8 \cdot 9 \cdot 10) = 10!.$$

So $10! - 7! \cdot 6! = 10! - 10! = 0$.

Thus, the correct answer is **B**.

3. For how many integer values of x is $|2x| \leq 7\pi$?

A 16

B 17

C 19

D 20

E 21

Solution:

The inequality $|2x| \leq 7\pi$ is equivalent to $|x| \leq \frac{7\pi}{2} \approx 10.99$. The integers satisfying this run from -10 to 10, which is $10 + 10 + 1 = 21$ values.

Thus, the correct answer is **E**.

4. Balls numbered $1, 2, 3, \ldots$ are deposited in 5 bins, labeled $A, B, C, D,$ and $E,$ using the following procedure. Ball 1 is deposited in bin $A,$ and balls 2 and 3 are deposited in $B.$ The next three balls are deposited in bin $C,$ the next 4 in bin $D,$ and so on, cycling back to bin A after balls are deposited in bin $E.$ (For example, $22, 23, \ldots, 28$ are deposited in bin B at step 7 of this process.) In which bin is ball 2024 deposited?

- A A
- B B
- C C
- D D
- E E

Solution:

Step k deposits k balls, so after step k a total of $\frac{k(k+1)}{2}$ balls have been placed.
Since $\frac{63 \cdot 64}{2} = 2016$ and $\frac{64 \cdot 65}{2} = 2080,$ ball 2024 falls in step $64.$
The steps cycle through the bins $A, B, C, D, E,$ so step k uses position $(k - 1) \bmod 5.$ Here $(64 - 1) \bmod 5 = 63 \bmod 5 = 3,$ which is the fourth bin, $D.$

Thus, the correct answer is **D**.

5. In the following expression, Melanie changed some of the plus signs to minus signs:

$$1+3+5+7+\cdots+97+99$$

When the new expression was evaluated, it was negative. What is the least number of plus signs that Melanie could have changed to minus signs?

- A 14
- B 15
- C 16
- D 17
- E 18

Solution:

The original expression sums the first 50 odd numbers, giving $50^2 = 2500$. Changing a term of value v from $+$ to $-$ decreases the total by $2v$, so to make the result negative the flipped terms must total more than $\frac{2500}{2} = 1250$.

To use as few terms as possible, flip the largest odd numbers $99, 97, 95, \ldots$ The largest k of them sum to $k(100-k)$. With $k = 14$ this is $14 \cdot 86 = 1204 \leq 1250$, but with $k = 15$ it is $15 \cdot 85 = 1275 > 1250$. So 15 sign changes suffice and 14 do not.

Thus, the correct answer is **B**.

6. The national debt of the United States is on track to reach $5 \cdot 10^{13}$ dollars by 2033. How many digits does this number of dollars have when written as a numeral in base 5? (The approximation of $\log_{10} 5$ as 0.7 is sufficient for this problem.)

A 18

B 20

C 22

D 24

E 26

Solution:

The number of digits of N in base 5 is $\lfloor \log_5 N \rfloor + 1$. With $N = 5 \cdot 10^{13}$,

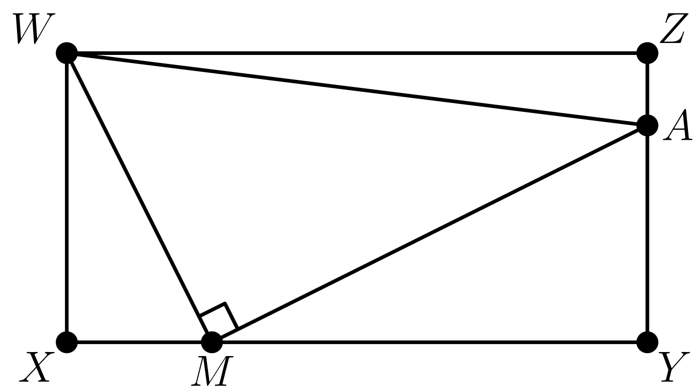
$$\log_{10} N = 13 + \log_{10} 5 = 13.7.$$

Converting bases, $\log_5 N = \dfrac{13.7}{\log_{10} 5} = \dfrac{13.7}{0.7} = 19.57\ldots$

Thus the number of digits is $\lfloor 19.57 \rfloor + 1 = 19 + 1 = 20.$

Thus, the correct answer is **B**.

7. In the figure below $WXYZ$ is a rectangle with $WX = 4$ and $WZ = 8$. Point M lies on $\overline{XY}$, point A lies on $\overline{YZ}$, and $\angle WMA$ is a right angle. The areas of $\triangle WXM$ and $\triangle WAZ$ are equal. What is the area of $\triangle WMA$?



A 13

B 14

C 15

D 16

E 17

Solution:

Set $X = (0,0), W = (0,4), Y = (8,0), Z = (8,4)$, with $M = (m,0)$ on $\overline{XY}$ and $A = (8,a)$ on $\overline{YZ}$.

Since $\angle WMA = 90^\circ$, $\overrightarrow{MW} \cdot \overrightarrow{MA} = (-m)(8-m) + 4a = 0$, so $4a = m(8-m)$. The areas give $[\triangle WXM] = \frac{1}{2} \cdot 4 \cdot m = 2m$ and $[\triangle WAZ] = \frac{1}{2} \cdot 8 \cdot (4-a) = 4(4-a)$. Setting these equal yields $m = 8 - 2a$.

Substituting $a = \frac{8-m}{2}$ into $4a = m(8-m)$ gives $2(8-m) = m(8-m)$, so $m = 2$ and $a = 3$. Then with $W = (0,4), M = (2,0), A = (8,3)$,

$$[\triangle WMA] = \frac{1}{2}\left|2(3-4) + 8(4-0)\right| = \frac{1}{2}(30) = 15.$$

Thus, the correct answer is **C**.

8. What value of x satisfies

$$\frac{\log_2 x \cdot \log_3 x}{\log_2 x + \log_3 x} = 2?$$

- A 25
- B 32
- C 36
- D 42
- E 48

Solution:

Dividing top and bottom by $\log_2 x \cdot \log_3 x$, the left side becomes

$$\frac{1}{\dfrac{1}{\log_2 x} + \dfrac{1}{\log_3 x}} = \frac{1}{\log_x 2 + \log_x 3} = \frac{1}{\log_x 6}.$$

So $\dfrac{1}{\log_x 6} = 2$, meaning $\log_x 6 = \dfrac{1}{2}$, i.e. $x^{1/2} = 6$.

Therefore $x = 36$.

Thus, the correct answer is **C**.

9. A dartboard is the region B in the coordinate plane consisting of points (x, y) such that $|x| + |y| \leq 8$. A target T is the region where $(x^2 + y^2 - 25)^2 \leq 49$. A dart is thrown and lands at a random point in B. The probability that the dart lands in T can be expressed as $\frac{m}{n} \cdot \pi$, where m and n are relatively prime positive integers. What is $m + n$?

- A 39
- B 71
- C 73
- D 75
- E 135

Solution:

The dartboard $|x| + |y| \leq 8$ is a square with diagonals 16, so its area is $\frac{1}{2} \cdot 16 \cdot 16 = 128$. The target condition $(x^2 + y^2 - 25)^2 \leq 49$ means $-7 \leq x^2 + y^2 - 25 \leq 7$, i.e. $18 \leq x^2 + y^2 \leq 32$, an annulus of area $\pi(32 - 18) = 14\pi$.

The distance from the origin to a side of the square (for instance $x + y = 8$) is $\frac{8}{\sqrt{2}} = \sqrt{32}$, exactly the annulus's outer radius. So the annulus is tangent to the square and lies entirely within B. The probability is $\frac{14\pi}{128} = \frac{7}{64}\pi$, giving $m + n = 7 + 64 = 71$.

Thus, the correct answer is **B**.

10. A list of 9 real numbers consists of $1, 2.2, 3.2, 5.2, 6.2,$ and $7,$ as well as x, y, z with $x \le y \le z.$ The range of the list is $7,$ and the mean and median are both positive integers. How many ordered triples (x, y, z) are possible?

- A 1
- B 2
- C 3
- D 4
- E infinitely many

Solution:

The six fixed numbers sum to $24.8.$ The mean $\dfrac{24.8 + x + y + z}{9}$ is an integer exactly when $x + y + z$ has fractional part $0.2.$ The fixed numbers span $[1, 7],$ so to make the range 7 the extremes must be pushed apart by one more unit.

Checking the possibilities gives exactly three valid triples:

$(x, y, z) = (0, 5, 6.2)$ with mean 4 and median $5;$ $(x, y, z) = (0.1, 4, 7.1)$ with mean 4 and median $4;$ and $(x, y, z) = (6, 6.2, 8)$ with mean 5 and median $6.$ Each has range 7 and integer mean and median.

Thus, the correct answer is **C**.

11. Let $x_n = \sin^2(n^\circ)$. What is the mean of $x_1, x_2, x_3, \ldots, x_{90}$?

A $\frac{11}{45}$

B $\frac{22}{45}$

C $\frac{89}{180}$

D $\frac{1}{2}$

E $\frac{91}{180}$

Solution:

Using $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$,

$$\sum_{n=1}^{90} \sin^2(n^\circ) = \frac{90}{2} - \frac{1}{2}\sum_{n=1}^{90} \cos(2n^\circ).$$

In the cosine sum, the terms for n and $90-n$ satisfy $\cos(2n^\circ) + \cos(180^\circ - 2n^\circ) = 0$, and $\cos 90^\circ = 0$, so everything cancels except $\cos 180^\circ = -1$.

Hence the sum is $45 - \frac{1}{2}(-1) = 45.5$, and the mean is $\dfrac{45.5}{90} = \dfrac{91}{180}$.

Thus, the correct answer is **E**.

12. Suppose z is a complex number with positive imaginary part, with real part greater than 1, and with $|z| = 2$. In the complex plane, the four points 0, z, z^2, and z^3 are the vertices of a quadrilateral with area 15. What is the imaginary part of z?

A $\frac{3}{4}$

B 1

C $\frac{4}{3}$

D $\frac{3}{2}$

E $\frac{5}{3}$

Solution:

For vertices $0, z, z^2, z^3$ the shoelace formula gives area

$$\tfrac{1}{2}\left|\mathrm{Im}(\bar{z}z^2 + \overline{z^2}\,z^3)\right| = \tfrac{1}{2}\left|\mathrm{Im}\big((|z|^2 + |z|^4)z\big)\right| = \tfrac{1}{2}(|z|^2 + |z|^4)\,\mathrm{Im}(z).$$

With $|z| = 2$, this is $\frac{1}{2}(4 + 16)\,\mathrm{Im}(z) = 10\,\mathrm{Im}(z)$. Setting $10\,\mathrm{Im}(z) = 15$ gives $\mathrm{Im}(z) = \dfrac{3}{2}$. (Then $\mathrm{Re}(z) = \sqrt{4 - \frac{9}{4}} = \frac{\sqrt{7}}{2} > 1$, as required.)

Thus, the correct answer is **D**.

13. There are real numbers $x, y, h,$ and k that satisfy the system of equations

$$x^2 + y^2 - 6x - 8y = h$$

$$x^2 + y^2 - 10x + 4y = k.$$

What is the minimum possible value of $h + k$?

A -54

B -46

C -34

D -16

E 16

Solution:

Adding the equations,

$$h + k = 2x^2 + 2y^2 - 16x - 4y = 2(x - 4)^2 + 2(y - 1)^2 - 34.$$

Both squared terms are nonnegative, so the minimum occurs at $x = 4, y = 1$, giving $h + k = -34$.

Thus, the correct answer is **C**.

14. How many different remainders can result when the 100th power of an integer is divided by 125?

A 1

B 2

C 5

D 25

E 125

Solution:

If n is coprime to 5, then since $\varphi(125) = 100$, Euler's theorem gives $n^{100} \equiv 1 \pmod{125}$. If n is a multiple of 5, then n^{100} is divisible by 5^{100}, hence by 125, leaving remainder 0.

So the only possible remainders are 0 and 1, which is 2 distinct values.

Thus, the correct answer is **B**.

15. A triangle in the coordinate plane has vertices $A(\log_2 1, \log_2 2), B(\log_2 3, \log_2 4)$, and $C(\log_2 7, \log_2 8)$. What is the area of $\triangle ABC$?

- A $\log_2 \frac{\sqrt{3}}{7}$
- B $\log_2 \frac{3}{\sqrt{7}}$
- C $\log_2 \frac{7}{\sqrt{3}}$
- D $\log_2 \frac{11}{\sqrt{7}}$
- E $\log_2 \frac{11}{\sqrt{3}}$

Solution:

The vertices are $A = (0, 1), B = (\log_2 3, 2), C = (\log_2 7, 3)$. By the shoelace formula,

$$[\triangle ABC] = \tfrac{1}{2}\left|0(2-3) + \log_2 3\,(3-1) + \log_2 7\,(1-2)\right| = \tfrac{1}{2}\left|2\log_2 3 - \log_2 7\right|.$$

This equals $\frac{1}{2}\log_2 \frac{9}{7} = \log_2 \sqrt{\frac{9}{7}} = \log_2 \frac{3}{\sqrt{7}}$.

Thus, the correct answer is **B**.

16. A group of 16 people will be partitioned into indistinguishable 4-person committees. Each committee will have one chairperson and one secretary. The number of different ways to make these assignments can be written as $3^r M$, where r and M are positive integers and M is not divisible by 3. What is r?

A 5

B 6

C 7

D 8

E 9

Solution:

The number of ways to split 16 people into 4 indistinguishable groups of 4 is $\frac{16!}{(4!)^4\, 4!}$. Each committee then chooses a chairperson and a secretary in $4 \cdot 3 = 12$ ways, contributing 12^4. So the total is $\frac{16!}{(4!)^4\, 4!} \cdot 12^4$.

Counting factors of $3 : 16!$ contributes $\lfloor 16/3 \rfloor + \lfloor 16/9 \rfloor = 6$. The denominator $(4!)^4\, 4!$ contributes $4 + 1 = 5$. And $12^4 = (2^2 \cdot 3)^4$ contributes 4. Thus $r = 6 - 5 + 4 = 5$.

Thus, the correct answer is **A**.

17. Integers a and b are randomly chosen without replacement from the set of integers with absolute value not exceeding 10. What is the probability that the polynomial $x^3 + ax^2 + bx + 6$ has 3 distinct integer roots?

A $\frac{1}{240}$

B $\frac{1}{221}$

C $\frac{1}{105}$

D $\frac{1}{84}$

E $\frac{1}{63}$

Solution:

The set has 21 integers, so there are $21 \cdot 20 = 420$ ordered choices of (a, b). If the polynomial has distinct integer roots p, q, r, then $pqr = -6, a = -(p + q + r)$, and $b = pq + qr + rp$.

The triples of distinct integers with product -6 are $\{1, 2, -3\}, \{1, -2, 3\}, \{-1, 2, 3\}$, $\{-1, -2, -3\}$, and $\{1, -1, 6\}$. These give $(a, b) = (0, -7), (-2, -5), (-4, 1)$, $(6, 11)$, and $(-6, -1)$. The fourth has $b = 11 > 10$, so it is invalid; the other four are valid and distinct.

The probability is $\frac{4}{420} = \frac{1}{105}$.

Thus, the correct answer is **C**.

18. The Fibonacci numbers are defined by $F_1 = 1, F_2 = 1,$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3.$ What is

$$\frac{F_2}{F_1} + \frac{F_4}{F_2} + \frac{F_6}{F_3} + \cdots + \frac{F_{20}}{F_{10}}?$$

- A 318
- B 319
- C 320
- D 321
- E 322

Solution:

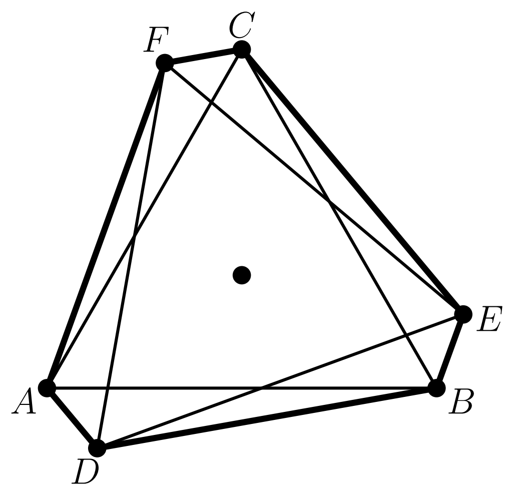
Since $F_{2k} = F_k L_k$ where L_k is the kth Lucas number, each term $\dfrac{F_{2k}}{F_k} = L_k.$ The sum is

$$L_1 + L_2 + \cdots + L_{10} = 1 + 3 + 4 + 7 + 11 + 18 + 29 + 47 + 76 + 123 = 319.$$

(Equivalently, $L_1 + \cdots + L_{10} = L_{12} - 3 = 322 - 3 = 319.$)

Thus, the correct answer is **B**.

19. Equilateral $\triangle ABC$ with side length 14 is rotated about its center by angle θ, where $0 < \theta < 60^\circ$, to form $\triangle DEF$. See the figure. The area of hexagon $ADBECF$ is $91\sqrt{3}$. What is $\tan\theta$?



A $\dfrac{3}{4}$

B $\dfrac{5\sqrt{3}}{11}$

C $\dfrac{4}{5}$

D $\dfrac{11}{13}$

E $\dfrac{7\sqrt{3}}{13}$

Solution:

The six vertices lie on the circumcircle of radius $R = \dfrac{14}{\sqrt{3}}$, so $R^2 = \dfrac{196}{3}$. Going around, the central angles alternate between θ (three times) and $120^\circ - \theta$ (three times). The cyclic-hexagon area is

$$\tfrac{1}{2}R^2\big(3\sin\theta + 3\sin(120^\circ - \theta)\big) = 98\big(\sin\theta + \sin(120^\circ - \theta)\big).$$

By sum-to-product, $\sin\theta + \sin(120^\circ - \theta) = 2\sin 60^\circ \cos(\theta - 60^\circ) = \sqrt{3}\cos(60^\circ - \theta)$. Setting the area to $91\sqrt{3}$ gives $\sqrt{3}\cos(60^\circ - \theta) = \dfrac{91\sqrt{3}}{98} = \dfrac{13\sqrt{3}}{14}$, so $\cos(60^\circ - \theta) = \dfrac{13}{14}$ and $\sin(60^\circ - \theta) = \dfrac{3\sqrt{3}}{14}$.

Then $\tan(60^\circ - \theta) = \dfrac{3\sqrt{3}}{13}$, and

$$\tan\theta = \tan\big(60^\circ - (60^\circ - \theta)\big) = \frac{\sqrt{3} - \frac{3\sqrt{3}}{13}}{1 + \sqrt{3} \cdot \frac{3\sqrt{3}}{13}} = \frac{\frac{10\sqrt{3}}{13}}{\frac{22}{13}} = \frac{5\sqrt{3}}{11}.$$

Thus, the correct answer is **B**.

20. Suppose A, B, and C are points in the plane with $AB = 40$ and $AC = 42$, and let x be the length of the line segment from A to the midpoint of $\overline{BC}$. Define a function f by letting $f(x)$ be the area of $\triangle ABC$. Then the domain of f is an open interval (p, q), and the maximum value r of $f(x)$ occurs at $x = s$. What is $p + q + r + s$?

- A 909
- B 910
- C 911
- D 912
- E 913

Solution:

Let $a = BC$. The median length gives $x^2 = \dfrac{2 \cdot 1600 + 2 \cdot 1764 - a^2}{4} = \dfrac{6728 - a^2}{4}$. The triangle inequality requires $2 < a < 82$, i.e. $4 < a^2 < 6724$, which translates to $1 < x < 41$. So $(p, q) = (1, 41)$.

With $AB = 40$ and $AC = 42$ fixed, the area $\frac{1}{2} \cdot 40 \cdot 42 \sin A$ is largest when $\angle A = 90^\circ$, giving $r = 840$. Then $a^2 = 40^2 + 42^2 = 3364$, so $x^2 = \dfrac{6728 - 3364}{4} = 841$, i.e. $s = 29$.

Thus $p + q + r + s = 1 + 41 + 840 + 29 = 911$.

Thus, the correct answer is **C**.

21. The measures of the smallest angles of three different right triangles sum to 90°. All three triangles have side lengths that are primitive Pythagorean triples. Two of them are 3-4-5 and 5-12-13. What is the perimeter of the third triangle?

A 40

B 126

C 154

D 176

E 208

Solution:

The smallest angles α, β of the 3-4-5 and 5-12-13 triangles have $\tan\alpha = \frac{3}{4}$ and $\tan\beta = \frac{5}{12}$. By the tangent addition formula,

$$\tan(\alpha+\beta) = \frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}} = \frac{\frac{14}{12}}{\frac{33}{48}} = \frac{56}{33}.$$

The third smallest angle γ satisfies $\gamma = 90^\circ - (\alpha+\beta)$, so $\tan\gamma = \dfrac{33}{56}$. The right triangle with legs 33 and 56 has hypotenuse $\sqrt{33^2+56^2} = \sqrt{4225} = 65$, a primitive triple. Its perimeter is $33+56+65 = 154.$

Thus, the correct answer is **C**.

22. Let $\triangle ABC$ be a triangle with integer side lengths and the property that $\angle B = 2\angle A$. What is the least possible perimeter of such a triangle?

A 13

B 14

C 15

D 16

E 17

Solution:

When $\angle B = 2\angle A$, the side lengths satisfy $b^2 = a(a+c)$, where $a = BC, b = CA,$ $c = AB$. So $c = \dfrac{b^2 - a^2}{a}$ must be a positive integer, and the sides must form a valid triangle.

Trying small values, $a = 4, b = 6$ gives $c = \dfrac{36 - 16}{4} = 5$, and the sides $4, 5, 6$ form a valid triangle with $6^2 = 4(4+5)$. Its perimeter is 15, and a search shows no smaller perimeter works.

Thus, the correct answer is **C**.

23. A right pyramid has regular octagon $ABCDEFGH$ with side length 1 as its base and apex V. Segments $\overline{AV}$ and $\overline{DV}$ are perpendicular. What is the square of the height of the pyramid?

A 1

B $\frac{1+\sqrt{2}}{2}$

C $\sqrt{2}$

D $\frac{3}{2}$

E $\frac{2+\sqrt{2}}{3}$

Solution:

Let R be the circumradius of the octagon and L the length of each lateral edge, so $L^2 = h^2 + R^2$. Since $\angle AVD = 90^\circ$, $AD^2 = 2L^2$.

Vertices A and D are three steps apart, a central angle of 135°, so $AD^2 = 2R^2(1 - \cos 135^\circ) = R^2(2+\sqrt{2})$. Setting $R^2(2+\sqrt{2}) = 2(h^2 + R^2)$ gives $2h^2 = R^2\sqrt{2}$.

For a regular octagon of side 1, $R^2 = \dfrac{1}{2\sin^2(22.5^\circ)} = \dfrac{2+\sqrt{2}}{2}$. Therefore $h^2 = \dfrac{R^2\sqrt{2}}{2} = \dfrac{(2+\sqrt{2})\sqrt{2}}{4} = \dfrac{2\sqrt{2}+2}{4} = \dfrac{1+\sqrt{2}}{2}$.

Thus, the correct answer is **B**.

24. What is the number of ordered triples (a, b, c) of positive integers, with $a \leq b \leq c \leq 9$, such that there exists a (non-degenerate) triangle $\triangle ABC$ with an integer inradius for which $a, b,$ and c are the lengths of the altitudes from A to $\overline{BC}$, B to $\overline{AC}$, and C to $\overline{AB}$, respectively? (Recall that the inradius of a triangle is the radius of the largest possible circle that can be inscribed in the triangle.)

A 2

B 3

C 4

D 5

E 6

Solution:

Writing each side as $\dfrac{2[\triangle]}{h}$, the semiperimeter is $[\triangle]\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$, so the inradius $r = \dfrac{[\triangle]}{s}$ satisfies $\dfrac{1}{r} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$. We need this to be $\dfrac{1}{r}$ for a positive integer r, with the sides (proportional to $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$) forming a non-degenerate triangle, requiring $\frac{1}{a} < \frac{1}{b} + \frac{1}{c}$.

Searching $a \leq b \leq c \leq 9$, the triples with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{r}$ that also satisfy the triangle inequality are exactly the equilateral ones: $(3, 3, 3)$ with $r = 1$, $(6, 6, 6)$ with $r = 2$, and $(9, 9, 9)$ with $r = 3$. Other solutions such as $(2, 3, 6)$ or $(4, 8, 8)$ give degenerate triangles. So there are 3 triples.

Thus, the correct answer is **B**.

25. Pablo will decorate each of 6 identical white balls with either a striped or a dotted pattern, using either red or blue paint. He will decide on the color and pattern for each ball by flipping a fair coin for each of the 12 decisions he must make. After the paint dries, he will place the 6 balls in an urn. Frida will randomly select one ball from the urn and note its color and pattern. The events "the ball Frida selects is red" and "the ball Frida selects is striped" may or may not be independent, depending on the outcome of Pablo's coin flips. The probability that these two events are independent can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is m? (Recall that two events A and B are independent if $P(A \text{ and } B) = P(A) \cdot P(B)$.)

- A 243
- B 245
- C 247
- D 249
- E 251

Solution:

Each ball is independently one of four equally likely types: red-striped, red-dotted, blue-striped, blue-dotted. Suppose among the 6 balls there are k red-striped, with R red and S striped in total. For Frida's uniform pick, $P(\text{red}) = \frac{R}{6}$, $P(\text{striped}) = \frac{S}{6}$, and $P(\text{red and striped}) = \frac{k}{6}$. Independence means $\frac{k}{6} = \frac{R}{6} \cdot \frac{S}{6}$, i.e. $6k = RS$.

Summing the multinomial counts of all type-assignments of the 6 balls satisfying $6k = RS$ gives 972 favorable outcomes out of $4^6 = 4096$. The probability is $\frac{972}{4096} = \frac{243}{1024}$, so $m = 243$.

Thus, the correct answer is **A**.

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