2024 AMC 12A Solutions

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1. What is the value of $9901 \cdot 101 - 99 \cdot 10101$?

A 2

B 20

C 21

D 200

E 2020

Solution:

Directly, $9901 \cdot 101 = 990100 + 9901 = 1000001$ and $99 \cdot 10101 = 1010100 - 10101 = 999999$. Their difference is $1000001 - 999999 = 2$. Thus, the correct answer is **A**.

2. A model used to estimate the time it will take to hike to the top of a mountain on a trail is of the form $T = aL + bG$, where a and b are constants, T is the time in minutes, L is the length of the trail in miles, and G is the altitude gain in feet. The model estimates that it will take 69 minutes to hike to the top if a trail is 1.5 miles long and ascends 800 feet, as well as if a trail is 1.2 miles long and ascends 1100 feet.

How many minutes does the model estimate it will take to hike to the top if the trail is 4.2 miles long and ascends 4000 feet?

- A 240
- B 246
- C 252
- D 258
- E 264

Solution:

From $1.5a + 800b = 1.2a + 1100b$ we get $0.3a = 300b$, so $a = 1000b$. Then $69 = 1.5a + 800b = 1500b + 800b = 2300b$, giving $b = 0.03$ and $a = 30$. For $L = 4.2,\ G = 4000, T = 30(4.2) + 0.03(4000) = 126 + 120 = 246$. Thus, the correct answer is **B**.

3. The number 2024 is written as the sum of not necessarily distinct two-digit numbers. What is the least number of two-digit numbers needed to write this sum?

- A 20
- B 21
- C 22
- D 23
- E 24

Solution:

Each number is at most 99, so k numbers sum to at most $99k$. Since $99 \cdot 20 = 1980 < 2024$, at least 21 numbers are required. With 21, we can use twenty 99s and one 44: $20 \cdot 99 + 44 = 1980 + 44 = 2024$. Thus, the correct answer is **B**.

4. What is the least value of n such that $n!$ is a multiple of 2024?

- A 11
- B 21
- C 22
- D 23
- E 253

Solution:

Factoring, $2024 = 2^3 \cdot 11 \cdot 23$. The factorial $n!$ contains the prime 23 only when $n \geq 23$. At $n = 23$, the product $23!$ already includes $23,\ 11$, and plenty of factors of 2, so $23!$ is a multiple of 2024. Thus, the correct answer is **D**.

5. A data set containing 20 numbers, some of which are 6, has mean 45. When all the 6s are removed, the data set has mean 66. How many 6s were in the original data set?

- A 4
- B 5
- C 6
- D 7
- E 8

Solution:

The full set sums to $20 \cdot 45 = 900$. Removing k sixes leaves $20 - k$ numbers summing to $900 - 6k$, with mean 66, so $900 - 6k = 66(20 - k) = 1320 - 66k$. Then $60k = 420$, giving $k = 7$. Thus, the correct answer is **D**.

6. The product of three integers is 60. What is the least possible positive sum of the three integers?

A 2

B 3

C 5

D 6

E 13

Solution:

To keep the product positive we use two negative integers $-p, -q$ and one positive r, with $pqr = 60$ and sum $r - p - q$. Trying $(p, q, r) = (1, 6, 10)$ gives product 60 and sum $10 - 1 - 6 = 3$. Checking the other factorizations shows no positive sum smaller than 3 is attainable (for example all-positive triples give at least $3 + 4 + 5 = 12$). Thus, the correct answer is **B**.

7. In $\triangle ABC$, $\angle ABC = 90^\circ$ and $BA = BC = \sqrt{2}$. Points $P_1, P_2, \ldots, P_{2024}$ lie on hypotenuse AC so that $AP_1 = P_1P_2 = P_2P_3 = \cdots = P_{2023}P_{2024} = P_{2024}C$.

What is the length of the vector sum

$$\vec{BP_1} + \vec{BP_2} + \vec{BP_3} + \cdots + \vec{BP_{2024}}?$$

- A 1011
- B 1012
- C 2023
- D 2024
- E 2025

Solution:

The points P_k are symmetric about the midpoint M of AC, so pairing P_k with its mirror gives $\vec{BP_k} + \vec{BP_{2025-k}} = 2\,\vec{BM}$. Hence the whole sum is $2024\,\vec{BM}$. In a right triangle the median to the hypotenuse has length half the hypotenuse; here $AC = 2$, so $BM = 1$. The length of the sum is $2024 \cdot 1 = 2024$. Thus, the correct answer is **D**.

8. How many angles θ with $0 \leq \theta \leq 2\pi$ satisfy $\log(\sin(3\theta)) + \log(\cos(2\theta)) = 0$?

A 0

B 1

C 2

D 3

E 4

Solution:

The equation means $\sin(3\theta)\cos(2\theta) = 1$ with both factors positive (for the logs to be defined). Since $\sin(3\theta) \leq 1$ and $\cos(2\theta) \leq 1$, their product is 1 only if $\sin(3\theta) = 1$ and $\cos(2\theta) = 1$ simultaneously. But $\cos(2\theta) = 1$ forces $\theta \in \{0, \pi, 2\pi\}$, where $\sin(3\theta) = 0 \neq 1$. No angle works. Thus, the correct answer is **A**.

9. Let M be the greatest integer such that both $M + 1213$ and $M + 3773$ are perfect squares. What is the units digit of M?

A 1

B 2

C 3

D 6

E 8

Solution:

Write $M + 1213 = a^2$ and $M + 3773 = b^2$, so $b^2 - a^2 = 2560$, i.e. $(b - a)(b + a) = 2560$. Both factors have the same parity, hence both even. To maximize a (and thus M), minimize $b - a$: take $b - a = 2,\ b + a = 1280$, so $a = 639$. Then $M = 639^2 - 1213 = 408321 - 1213 = 407108$, whose units digit is 8. Thus, the correct answer is **E**.

10. Let α be the radian measure of the smallest angle in a 3-4-5 right triangle. Let β be the radian measure of the smallest angle in a 7-24-25 right triangle. In terms of α, what is β?

A $\frac{\alpha}{3}$

B $\alpha - \frac{\pi}{8}$

C $\frac{\pi}{2} - 2\alpha$

D $\frac{\alpha}{2}$

E $\pi - 4\alpha$

Solution:

The smallest angle of the 3-4-5 triangle has $\tan\alpha = \frac{3}{4}$. Then

$$\tan 2\alpha = \frac{2 \cdot \frac{3}{4}}{1 - \frac{9}{16}} = \frac{3/2}{7/16} = \frac{24}{7}.$$

The smallest angle of the 7-24-25 triangle has $\tan\beta = \frac{7}{24} = \cot 2\alpha = \tan\left(\frac{\pi}{2} - 2\alpha\right)$. Hence $\beta = \frac{\pi}{2} - 2\alpha$. Thus, the correct answer is **C**.

11. There are exactly K positive integers b with $5 \le b \le 2024$ such that the base-b integer 2024_b is divisible by 16 (where 16 is in base ten). What is the sum of the digits of K?

A 16

B 17

C 18

D 20

E 21

Solution:

Here $2024_b = 2b^3 + 2b + 4 = 2(b^3 + b + 2)$, so $16 \mid 2024_b$ exactly when $8 \mid b^3 + b + 2$. Checking residues $\mod 8$, $b^3 + b + 2 \equiv 0$ precisely for $b \equiv 3, 6, 7 \pmod 8$.

Counting b in $[5, 2024]$: residue 3 gives $11, \ldots, 2019$ (252 values), residue 6 gives $6, \ldots, 2022$ (253 values), and residue 7 gives $7, \ldots, 2023$ (253 values). So $K = 252 + 253 + 253 = 758$, and its digit sum is $7 + 5 + 8 = 20$.

Thus, the correct answer is **D**.

12. The first three terms of a geometric sequence are the integers $a,\ 720,$ and b, where $a < 720 < b$. What is the sum of the digits of the least possible value of b?

A 9

B 12

C 16

D 18

E 21

Solution:

Since the terms are geometric, $720^2 = ab$, so $ab = 518400 = 2^8 \cdot 3^4 \cdot 5^2$. Because $b = 518400/a$, minimizing b means maximizing the divisor a subject to $a < 720$. The largest such divisor is $a = 675 = 3^3 \cdot 5^2$, giving $b = 518400/675 = 768$. Its digit sum is $7 + 6 + 8 = 21$. Thus, the correct answer is **E**.

13. The graph of $y = e^{x+1} + e^{-x} - 2$ has an axis of symmetry. What is the reflection of the point $\left(-1, \frac{1}{2}\right)$ over this axis?

A $\left(-1, -\frac{3}{2}\right)$

B $(-1, 0)$

C $\left(-1, \frac{1}{2}\right)$

D $\left(0, \frac{1}{2}\right)$

E $\left(3, \frac{1}{2}\right)$

Solution:

The curve $y = e^{x+1} + e^{-x} - 2$ is symmetric about the vertical line through its minimum. Setting the derivative $e^{x+1} - e^{-x} = 0$ gives $x + 1 = -x$, so $x = -\frac{1}{2}$. Reflecting $\left(-1, \frac{1}{2}\right)$ across $x = -\frac{1}{2}$ keeps the y-coordinate and sends $x = -1$ to $x = 0$. The image is $\left(0, \frac{1}{2}\right)$. Thus, the correct answer is **D**.

14. The numbers, in order, of each row and the numbers, in order, of each column of a 5×5 array of integers form an arithmetic progression of length 5. The numbers in positions $(5,5),\ (2,4),\ (4,3)$, and $(3,1)$ are $0,\ 48,\ 16$, and 12, respectively. What number is in position $(1,2)$?

$$\begin{bmatrix} \cdot & ? & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 48 & \cdot \\ 12 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 16 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}$$

A 19

B 24

C 29

D 34

E 39

Solution:

A grid whose rows and columns are all arithmetic has entries of the bilinear form $a(i,j) = \alpha + \beta i + \gamma j + \delta ij$. The four givens yield

$$\alpha + 5\beta + 5\gamma + 25\delta = 0, \quad \alpha + 2\beta + 4\gamma + 8\delta = 48,$$

$$\alpha + 4\beta + 3\gamma + 12\delta = 16, \quad \alpha + 3\beta + \gamma + 3\delta = 12.$$

Solving gives $\delta = -5,\ \beta = 5,\ \gamma = 22,\ \alpha = -10$. Then $a(1,2) = \alpha + \beta + 2\gamma + 2\delta = -10 + 5 + 44 - 10 = 29$.

Thus, the correct answer is **C**.

15. The roots of x^3+2x^2-x+3 are $p,\ q$, and r. What is the value of

$$(p^2+4)(q^2+4)(r^2+4)?$$

- A 64
- B 75
- C 100
- D 125
- E 144

Solution:

Since $P(x)=(x-p)(x-q)(x-r)$, grouping $p^2+4=(p-2i)(p+2i)$ over all roots gives

$$\prod(p^2+4)=P(2i)\,P(-2i).$$

Compute $P(2i)=-8i-8-2i+3=-5-10i$ and $P(-2i)=8i-8+2i+3=-5+10i$. Their product is $(-5)^2+10^2=25+100=125$. Thus, the correct answer is **D**.

16. A set of 12 tokens — 3 red, 2 white, 1 blue, and 6 black — is to be distributed at random to 3 game players, 4 tokens per player. The probability that some player gets all the red tokens, another gets all the white tokens, and the remaining player gets the blue token can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

- A 387
- B 388
- C 389
- D 390
- E 391

Solution:

Treat all tokens as distinct; the total number of ways to deal 4 to each player is $\binom{12}{4,4,4} = 34650$. For the favorable event, choose which player gets the reds, whites, and blue in $3! = 6$ ways. The red player needs 1 more token, the white player 2 more, and the blue player 3 more, all black; the 6 black tokens split as $1, 2, 3$ in $\frac{6!}{1!\,2!\,3!} = 60$ ways. So the probability is $\frac{6 \cdot 60}{34650} = \frac{360}{34650} = \frac{4}{385}$. Then $m+n = 4 + 385 = 389$. Thus, the correct answer is **C**.

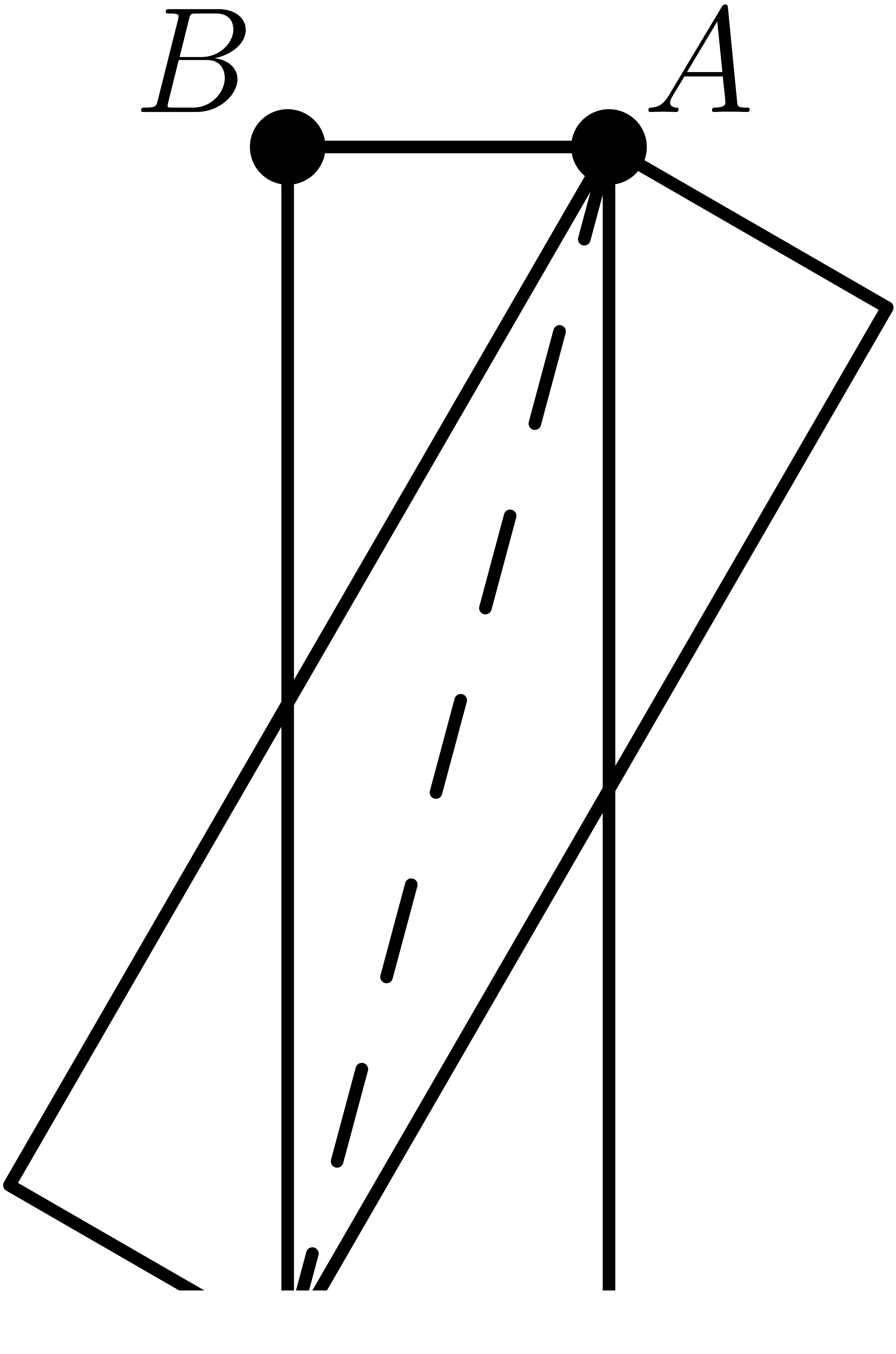
17. Integers $a,\ b$, and c satisfy $ab + c = 100$, $bc + a = 87$, and $ca + b = 60$. What is $ab + bc + ca$?

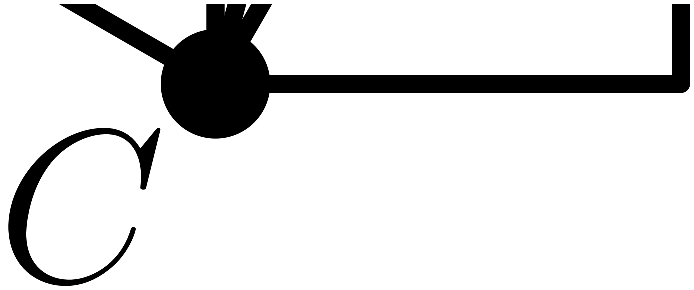
- A 212
- B 247
- C 258
- D 276
- E 284

Solution:

Subtracting pairs gives $(a - c)(b - 1) = 13,\ (a - b)(c - 1) = -27$, and $(b - c)(a - 1) = 40$. Because 13 is prime, only a few cases arise; testing them yields $a = -9,\ b = -12,\ c = -8$, which satisfy all three original equations. Then $ab + bc + ca = 108 + 96 + 72 = 276$. Thus, the correct answer is **D**.

18. On top of a rectangular card with sides of length 1 and $2+\sqrt{3}$, an identical card is placed so that two of their diagonals line up, as shown (AC, in this case).


B
A



Continue the process, adding a third card to the second, and so on, lining up successive diagonals after rotating clockwise. In total, how many cards must be used until a vertex of a new card lands exactly on the vertex labeled B in the figure?

A 6

B 8

C 10

D 12

E No new vertex will land on B.

Solution:

The diagonal of the card makes an angle θ with the long side where $\tan\theta = \dfrac{1}{2+\sqrt{3}} = 2-\sqrt{3} = \tan 15^\circ$, so $\theta = 15^\circ$. All the cards share diagonals that are equal chords (diameters) of one common circle, and each newly added card is the previous one turned 15° clockwise about the common center. A fresh vertex first coincides with B once the accumulated rotation reaches 90°, i.e. after $90^\circ/15^\circ = 6$ cards. Since 15° divides 90° evenly, a vertex does land on B. Thus, the correct answer is **A**.

19. Cyclic quadrilateral $ABCD$ has lengths $BC = CD = 3$ and $DA = 5$ with $\angle CDA = 120^\circ$. What is the length of the shorter diagonal of $ABCD$?

A $\frac{31}{7}$

B $\frac{33}{7}$

C 5

D $\frac{39}{7}$

E $\frac{41}{7}$

Solution:

In $\triangle ACD$, the law of cosines gives $AC^2 = 9 + 25 - 2(15)\cos 120^\circ = 34 + 15 = 49$, so $AC = 7$.

Since $ABCD$ is cyclic, $\angle ABC = 180^\circ - 120^\circ = 60^\circ$. In $\triangle ABC$ with $BC = 3$ and $AC = 7$, the law of cosines gives $49 = AB^2 + 9 - 3AB$, so $AB = 8$. By Ptolemy, $AC \cdot BD = AB \cdot CD + BC \cdot DA = 8 \cdot 3 + 3 \cdot 5 = 39$, hence $BD = \frac{39}{7}$. This is shorter than $AC = 7$.

Thus, the correct answer is **D**.

20. Points P and Q are chosen uniformly and independently at random on sides $\overline{AB}$ and $\overline{AC}$, respectively, of equilateral triangle $\triangle ABC$. Which of the following intervals contains the probability that the area of $\triangle APQ$ is less than half the area of $\triangle ABC$?

A $\left[\frac{3}{8}, \frac{1}{2}\right]$

B $\left(\frac{1}{2}, \frac{2}{3}\right]$

C $\left(\frac{2}{3}, \frac{3}{4}\right]$

D $\left(\frac{3}{4}, \frac{7}{8}\right]$

E $\left(\frac{7}{8}, 1\right]$

Solution:

With $x = \frac{AP}{AB}$ and $y = \frac{AQ}{AC}$ uniform on $[0, 1]$, the area ratio $\frac{[APQ]}{[ABC]} = xy$. The complementary event $xy \geq \frac{1}{2}$ requires $x \geq \frac{1}{2}$ and $y \in [\frac{1}{2x}, 1]$, with probability

$$\int_{1/2}^{1} \left(1 - \frac{1}{2x}\right) dx = \frac{1}{2} - \frac{\ln 2}{2} \approx 0.153.$$

Therefore $P(xy < \frac{1}{2}) \approx 1 - 0.153 = 0.847$, which lies in $\left(\frac{3}{4}, \frac{7}{8}\right]$. Thus, the correct answer is **D**.

21. Suppose that $a_1 = 2$ and the sequence (a_n) satisfies the recurrence relation

$$\frac{a_n - 1}{n - 1} = \frac{a_{n-1} + 1}{(n-1) + 1}$$

for all $n \geq 2$. What is the greatest integer less than or equal to

$$\sum_{n=1}^{100} a_n^2?$$

A $338{,}550$

B $338{,}551$

C $338{,}552$

D $338{,}553$

E $338{,}554$

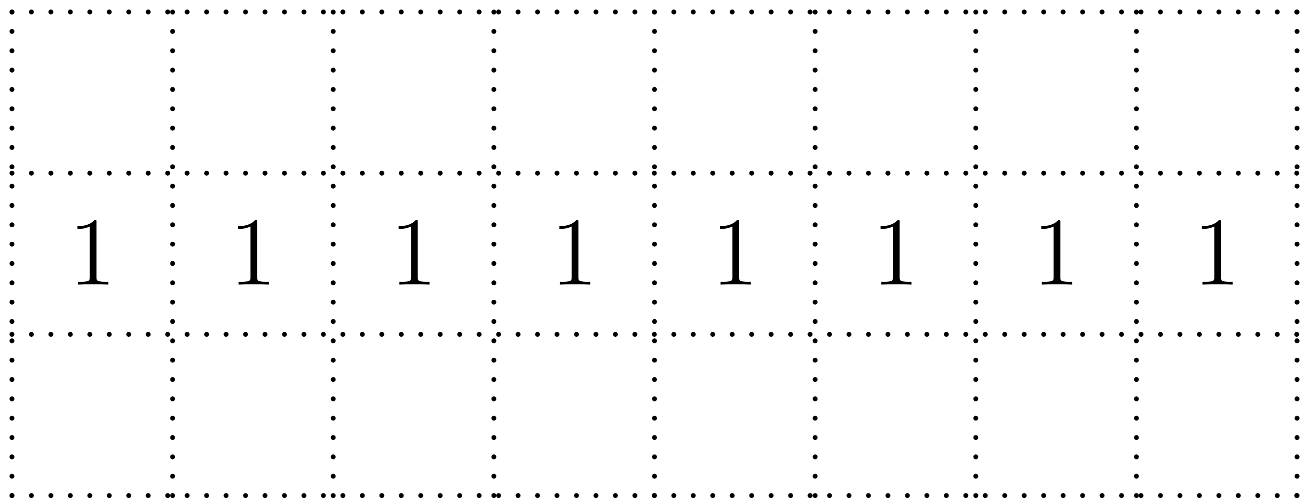
Solution:

The recurrence rearranges to $a_n = 1 + \frac{n-1}{n}(a_{n-1} + 1)$. Computing early terms $2, \frac{5}{2}, \frac{10}{3}, \frac{17}{4}, \ldots$ reveals $a_n = n + \frac{1}{n}$, which checks out. Then $a_n^2 = n^2 + 2 + \frac{1}{n^2}$, so

$$\sum_{n=1}^{100} a_n^2 = \sum_{n=1}^{100} n^2 + 200 + \sum_{n=1}^{100} \frac{1}{n^2} = 338350 + 200 + S,$$

where $1 < S < 2$. Hence the sum is between 338550 and 338552, and its floor is 338551. Thus, the correct answer is **B**.

22. The figure below shows a dotted grid 8 cells wide and 3 cells tall consisting of $1'' \times 1''$ squares. Carl places 1-inch toothpicks along some of the sides of the squares to create a closed loop that does not intersect itself. The numbers in the cells indicate the number of sides of that square that are to be covered by toothpicks, and any number of toothpicks are allowed if no number is written. In how many ways can Carl place the toothpicks?



- A 130
- B 144
- C 146
- D 162
- E 196

Solution:

The toothpicks form a single simple closed curve. The constraint forces each of the eight middle-row cells to have exactly one of its four sides used, which severely limits how the loop threads through the grid: for each middle cell the loop must contribute one edge (a top, bottom, left, or right side), and consecutive choices must join up into one non-self-intersecting closed curve.

Working left to right and tracking how the loop's upper and lower portions enter and leave each column (equivalently, a transfer-matrix/casework count over the 8

columns) enumerates all admissible loops. Carrying out this casework gives 146 valid configurations.

Thus, the correct answer is **C**.

23. What is the value of

$$\tan^2\frac{\pi}{16}\cdot\tan^2\frac{3\pi}{16}+\tan^2\frac{\pi}{16}\cdot\tan^2\frac{5\pi}{16}+\tan^2\frac{3\pi}{16}\cdot\tan^2\frac{7\pi}{16}+\tan^2\frac{5\pi}{16}\cdot\tan^2\frac{7\pi}{16}?$$

- A 28
- B 68
- C 70
- D 72
- E 84

Solution:

With $a=\tan^2\frac{\pi}{16},\ b=\tan^2\frac{3\pi}{16},\ c=\tan^2\frac{5\pi}{16},\ d=\tan^2\frac{7\pi}{16}$, the expression is $ab+ac+bd+cd=(a+d)(b+c)$.

Since $\frac{7\pi}{16}=\frac{\pi}{2}-\frac{\pi}{16}$, we have $d=\cot^2\frac{\pi}{16}$, so $a+d=\tan^2\frac{\pi}{16}+\cot^2\frac{\pi}{16}=\frac{4}{\sin^2(\pi/8)}-2=14+8\sqrt{2}$. Likewise $b+c=\frac{4}{\sin^2(3\pi/8)}-2=14-8\sqrt{2}$. Their product is $14^2-(8\sqrt{2})^2=196-128=68$.

Thus, the correct answer is **B**.

24. A *disphenoid* is a tetrahedron whose triangular faces are congruent to one another. What is the least total surface area of a disphenoid whose faces are scalene triangles with integer side lengths?

A $\sqrt{3}$

B $3\sqrt{15}$

C 15

D $15\sqrt{7}$

E $24\sqrt{6}$

Solution:

A disphenoid exists (as the tetrahedron formed by the face-plane midpoints of a box) exactly when the common face triangle is acute, and its total surface area is 4 times one face's area. We want the smallest-area acute scalene integer triangle. The candidates $(2,3,4)$ and $(3,5,6)$ are obtuse, and $(3,4,5)$ is right (giving a degenerate flat figure), but $(4,5,6)$ is acute since $4^2+5^2>6^2$. By Heron with $s=\frac{15}{2}$, its area is $\sqrt{\frac{15}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}}=\frac{15\sqrt{7}}{4}$. The total surface area is $4\cdot\frac{15\sqrt{7}}{4}=15\sqrt{7}$. Thus, the correct answer is **D**.

25. A graph is *symmetric* about a line if the graph remains unchanged after reflection in that line. For how many quadruples of integers (a, b, c, d), where $|a|, |b|, |c|, |d| \leq 5$ and c and d are not both 0, is the graph of

$$y = \frac{ax + b}{cx + d}$$

symmetric about the line $y = x$?

A 1282

B 1292

C 1310

D 1320

E 1330

Solution:

Reflecting the graph of $y = f(x)$ over $y = x$ produces the graph of its inverse, so the graph is symmetric about $y = x$ exactly when f equals its own inverse. For $f(x) = \frac{ax+b}{cx+d}$ this happens in two ways: when $a + d = 0$ with $ad - bc \neq 0$ (a genuine involution, including the slope-1 lines when $c = 0$), or when f is the identity $y = x$ ($b = c = 0,\ a = d \neq 0$).

For $a + d = 0$, set $d = -a$; the determinant $-a^2 - bc$ must be nonzero, so we need $a^2 + bc \neq 0$, together with $(c, d) \neq (0, 0)$. Counting (a, b, c) with each in $[-5, 5]$ gives 1282 quadruples. The identity case adds 10 more ($a = d \in \{\pm 1, \ldots, \pm 5\}$). The total is $1282 + 10 = 1292$.

Thus, the correct answer is **B**.

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