2023 AMC 12B Solutions

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1. Mrs. Jones is pouring orange juice into four identical glasses for her four sons. She fills the first three glasses completely but runs out of juice when the fourth glass is only $\frac{1}{3}$ full. What fraction of a glass must Mrs. Jones pour from each of the first three glasses into the fourth glass so that all four glasses will have the same amount of juice?

A $\dfrac{1}{12}$

B $\dfrac{1}{4}$

C $\dfrac{1}{6}$

D $\dfrac{1}{8}$

E $\dfrac{2}{9}$

Solution:

The total juice is $3+\frac{1}{3}=\frac{10}{3}$ glasses. Split evenly, each glass should hold $\frac{10}{3}\div 4=\frac{5}{6}$ of a glass. Each of the first three glasses must therefore give up $1-\frac{5}{6}=\frac{1}{6}$.

Thus, the correct answer is **C**.

2. Carlos went to a sports store to buy running shoes. Running shoes were on sale, with prices reduced by 20% on every pair of shoes. Carlos also knew that he had to pay a 7.5% sales tax on the discounted price. He had 43 dollars. What is the original (before discount) price of the most expensive shoes he could afford to buy?

- A 46
- B 50
- C 48
- D 47
- E 49

Solution:

The final cost of a pair with original price p is $0.8 \times 1.075 \times p = 0.86p$. Setting $0.86p \leq 43$ gives $p \leq 50$, so the most expensive affordable pair originally cost 50 dollars.

Thus, the correct answer is **B**.

3. A 3-4-5 right triangle is inscribed in circle A, and a 5-12-13 right triangle is inscribed in circle B. What is the ratio of the area of circle A to the area of circle B?

A $\frac{9}{25}$

B $\frac{1}{9}$

C $\frac{1}{5}$

D $\frac{25}{169}$

E $\frac{4}{25}$

Solution:

The hypotenuse of an inscribed right triangle is a diameter, so circle A has diameter 5 and circle B has diameter 13. The ratio of areas is $\left(\frac{5}{13}\right)^2 = \frac{25}{169}$.

Thus, the correct answer is **D**.

4. Jackson's paintbrush makes a narrow strip with a width of 6.5 millimeters. Jackson has enough paint to make a strip 25 meters long. How many square centimeters of paper could Jackson cover with paint?

A $162{,}500$

B 162.5

C $1{,}625$

D $1{,}625{,}000$

E $16{,}250$

Solution:

Converting units, the strip is 0.65 cm wide and 2500 cm long, so its area is $0.65 \times 2500 = 1625$ square centimeters.

Thus, the correct answer is **C**.

5. You are playing a game. A 2×1 rectangle covers two adjacent squares (oriented either horizontally or vertically) of a 3×3 grid of squares, but you are not told which two squares are covered. Your goal is to find at least one square that is covered by the rectangle. A "turn" consists of you guessing a square, after which you are told whether that square is covered by the hidden rectangle. What is the minimum number of turns you need to ensure that at least one of your guessed squares is covered by the rectangle?

- A 3
- B 5
- C 4
- D 8
- E 6

Solution:

A set of guessed squares is guaranteed to hit the domino if and only if the un-guessed squares contain no two adjacent squares, since otherwise the domino could hide on that adjacent pair. The largest set of pairwise non-adjacent squares in the 3×3 grid is the 5-square checkerboard (four corners plus the center). So at most 5 squares can be left unguessed, and you must guess $9 - 5 = 4$.

Thus, the correct answer is **C**.

6. When the roots of the polynomial

$$P(x) = (x-1)^1(x-2)^2(x-3)^3 \cdots (x-10)^{10}$$

are removed from the number line, what remains is the union of 11 disjoint open intervals. On how many of these intervals is $P(x)$ positive?

- A 3
- B 7
- C 6
- D 4
- E 5

Solution:

The exponent of the factor $(x-k)$ is k, so the sign of P changes at $x = k$ only when k is odd, i.e. at $1, 3, 5, 7, 9$. For $x > 10$ every factor is positive, so $P > 0$. Sweeping left and flipping at each odd root, the positive intervals are $(10, \infty)$, $(9, 10)$, $(6, 7)$, $(5, 6)$, $(2, 3)$, and $(1, 2)$ – six intervals in all.

Thus, the correct answer is **C**.

7. For how many integers n does the expression

$$\sqrt{\frac{\log(n^2) - (\log n)^2}{\log n - 3}}$$

represent a real number, where $\log$ denotes the base 10 logarithm?

A 900

B 3

C 902

D 2

E 901

Solution:

Write $L = \log n$. Then $\log(n^2) - (\log n)^2 = 2L - L^2 = L(2 - L)$, and the fraction is $\dfrac{L(2-L)}{L-3}$. A sign chart shows this is ≥ 0 exactly when $L \leq 0$ or $2 \leq L < 3$. Since n is a positive integer, $L \leq 0$ forces $n = 1$, while $2 \leq L < 3$ gives $100 \leq n \leq 999$, which is 900 values. In total $1 + 900 = 901$.

Thus, the correct answer is **E**.

8. How many nonempty subsets B of $\{0, 1, 2, 3, \ldots, 12\}$ have the property that the number of elements in B is equal to the least element of B? For example, $B = \{4, 6, 8, 11\}$ satisfies the condition.

- A 256
- B 136
- C 108
- D 144
- E 156

Solution:

If the least element is m $(m \geq 1)$, then $|B| = m$ and the remaining $m - 1$ elements come from $\{m + 1, \ldots, 12\}$, a set of size $12 - m$. The count is

$$\sum_{m \geq 1} \binom{12 - m}{m - 1} = \binom{11}{0} + \binom{10}{1} + \binom{9}{2} + \binom{8}{3} + \binom{7}{4} + \binom{6}{5},$$

which equals $1 + 10 + 36 + 56 + 35 + 6 = 144.$

Thus, the correct answer is **D**.

9. What is the area of the region in the coordinate plane defined by

$$||x| - 1| + ||y| - 1| \leq 1?$$

A 2

B 8

C 4

D 15

E 12

Solution:

Replacing x, y by $|x|, |y|$, the condition $|x - 1| + |y - 1| \leq 1$ describes a diamond centered at $(1, 1)$ with diagonals of length 2, hence area 2. It lies entirely in the first quadrant (touching the axes only at single points), so reflecting across the two axes produces 4 disjoint copies. The total area is $4 \times 2 = 8$.

Thus, the correct answer is **B**.

10. In the xy-plane, a circle of radius 4 with center on the positive x-axis is tangent to the y-axis at the origin, and a circle with radius 10 with center on the positive y-axis is tangent to the x-axis at the origin. What is the slope of the line passing through the two points at which these circles intersect?

A $\frac{2}{7}$

B $\frac{3}{7}$

C $\frac{2}{\sqrt{29}}$

D $\frac{1}{\sqrt{29}}$

E $\frac{2}{5}$

Solution:

The circles are $(x-4)^2+y^2=16$ and $x^2+(y-10)^2=100$, i.e. $x^2+y^2=8x$ and $x^2+y^2=20y$. Subtracting gives $8x=20y$, so the intersection points lie on $y=\frac{2}{5}x$, which has slope $\frac{2}{5}$.

Thus, the correct answer is **E**.

11. What is the maximum area of an isosceles trapezoid that has legs of length 1 and one base twice as long as the other?

A $\frac{5}{4}$

B $\frac{8}{7}$

C $\frac{5\sqrt{2}}{4}$

D $\frac{3}{2}$

E $\frac{3\sqrt{3}}{4}$

Solution:

Let the bases be a and $2a$. Each leg has horizontal offset $\frac{a}{2}$, so the height is $\sqrt{1-\frac{a^2}{4}}$ and the area is $A=\frac{3a}{2}\sqrt{1-\frac{a^2}{4}}$. Then $A^2=\frac{9}{4}\left(a^2-\frac{a^4}{4}\right)$, maximized when $a^2=2$. There the height is $\frac{1}{\sqrt{2}}$ and $A=\frac{3\sqrt{2}}{2}\cdot\frac{1}{\sqrt{2}}=\frac{3}{2}$.

Thus, the correct answer is **D**.

12. For complex numbers $u = a + bi$ and $v = c + di$, define the binary operation $\otimes$ by

$$u \otimes v = ac + bdi.$$

Suppose z is a complex number such that $z \otimes z = z^2 + 40$. What is $|z|$?

A 2

B 5

C $\sqrt{5}$

D $\sqrt{10}$

E $5\sqrt{2}$

Solution:

With $z = a + bi$, we have $z \otimes z = a^2 + b^2 i$ and $z^2 + 40 = (a^2 - b^2 + 40) + 2abi$. The real parts give $a^2 = a^2 - b^2 + 40$, so $b^2 = 40$. The imaginary parts give $b^2 = 2ab$, so $b = 2a$ and $a^2 = \frac{b^2}{4} = 10$. Then $|z|^2 = a^2 + b^2 = 50$, so $|z| = 5\sqrt{2}$.

Thus, the correct answer is **E**.

13. A rectangular box P has distinct edge lengths $a, b,$ and c. The sum of the lengths of all 12 edges of P is $13,$ the sum of the areas of all 6 faces of P is $\frac{11}{2}$, and the volume of P is $\frac{1}{2}$. What is the length of the longest interior diagonal connecting two vertices of P?

A 2

B $\frac{3}{8}$

C $\frac{9}{8}$

D $\frac{9}{4}$

E $\frac{3}{2}$

Solution:

From the edges, $4(a+b+c) = 13,$ so $a+b+c = \frac{13}{4}$. From the faces, $2(ab+bc+ca) = \frac{11}{2}$, so $ab+bc+ca = \frac{11}{4}$. Then

$$a^2+b^2+c^2 = \left(\frac{13}{4}\right)^2 - 2\cdot\frac{11}{4} = \frac{169}{16} - \frac{88}{16} = \frac{81}{16},$$

so the diagonal is $\sqrt{\frac{81}{16}} = \frac{9}{4}$.

Thus, the correct answer is **D**.

14. For how many ordered pairs (a, b) of integers does the polynomial $x^3 + ax^2 + bx + 6$ have 3 distinct integer roots?

A 5

B 6

C 8

D 7

E 4

Solution:

By Vieta, the three distinct integer roots multiply to -6. The sets of three distinct integers with product -6 are $\{1, 2, -3\}, \{1, -2, 3\}, \{-1, 2, 3\}, \{-1, -2, -3\}$, and $\{1, -1, 6\}$. Each set determines $a = -(p + q + r)$ and $b = pq + qr + rp$, and all five give different pairs, so there are 5 ordered pairs (a, b).

Thus, the correct answer is **A**.

15. Suppose $a, b,$ and c are positive integers such that

$$\frac{a}{14} + \frac{b}{15} = \frac{c}{210}.$$

Which of the following statements are necessarily true?

I. If $\gcd(a, 14) = 1$ or $\gcd(b, 15) = 1$ or both, then $\gcd(c, 210) = 1.$

II. If $\gcd(c, 210) = 1,$ then $\gcd(a, 14) = 1$ or $\gcd(b, 15) = 1$ or both.

III. $\gcd(c, 210) = 1$ if and only if $\gcd(a, 14) = \gcd(b, 15) = 1.$

A I, II, and III

B I only

C I and II only

D III only

E II and III only

Solution:

Multiplying by 210 gives $c = 15a + 14b.$ Since $15 \equiv 1 \pmod{14},$ we get $c \equiv a \pmod{14},$ so $\gcd(c, 14) = 1$ iff $\gcd(a, 14) = 1.$ Since $14 \equiv -1 \pmod{15},$ we get $c \equiv -b \pmod{15},$ so $\gcd(c, 15) = 1$ iff $\gcd(b, 15) = 1.$ As $210 = 14 \cdot 15$ with $\gcd(14, 15) = 1,$ statement III follows: $\gcd(c, 210) = 1$ iff both hold. Statement II is the forward implication of III, hence true. Statement I is false: if $\gcd(a, 14) = 1$ but $\gcd(b, 15) \neq 1,$ then $\gcd(c, 15) \neq 1,$ so $\gcd(c, 210) \neq 1.$ Only II and III are true.

Thus, the correct answer is **E**.

16. In Coinland, there are three types of coins, each worth $6, 10,$ and $15.$ What is the sum of the digits of the maximum amount of money that is impossible to have?

A 8

B 10

C 7

D 11

E 9

Solution:

The amounts $30, 31, 32, 33, 34, 35$ are all attainable (for instance $30 = 6 \cdot 5, 31 = 6 + 10 + 15, 32 = 6 \cdot 2 + 10 \cdot 2, 33 = 6 \cdot 3 + 15, 34 = 6 \cdot 4 + 10, 35 = 10 + 10 + 15$). Adding 6's then reaches every larger amount. Checking below, 29 is impossible, since $29 - 6,\ 29 - 10,\ 29 - 15$ are all impossible. So the largest impossible amount is $29,$ whose digit sum is $2 + 9 = 11.$

Thus, the correct answer is **D**.

17. Triangle ABC has side lengths in arithmetic progression, and the smallest side has length $6.$ If the triangle has an angle of $120°$, what is the area of ABC?

A $12\sqrt{3}$

B $8\sqrt{6}$

C $14\sqrt{2}$

D $20\sqrt{2}$

E $15\sqrt{3}$

Solution:

Let the sides be $6,\ 6+d,\ 6+2d$. The 120° angle faces the longest side, so $(6+2d)^2 = 6^2 + (6+d)^2 - 2 \cdot 6 \cdot (6+d)\cos 120^\circ$. Using $\cos 120^\circ = -\frac{1}{2}$ gives $3d^2 + 6d - 72 = 0$, so $d = 4$ and the sides are $6, 10, 14$. The area is $\frac{1}{2} \cdot 6 \cdot 10 \cdot \sin 120^\circ = 30 \cdot \frac{\sqrt{3}}{2} = 15\sqrt{3}$.

Thus, the correct answer is **E**.

18. Last academic year Yolanda and Zelda took different courses that did not necessarily administer the same number of quizzes during each of the two semesters. Yolanda's average on all the quizzes she took during the first semester was 3 points higher than Zelda's average on all the quizzes she took during the first semester. Yolanda's average on all the quizzes she took during the second semester was 18 points higher than her average for the first semester and was again 3 points higher than Zelda's average on all the quizzes Zelda took during her second semester. Which one of the following statements cannot possibly be true?

- A Yolanda's quiz average for the academic year was 22 points higher than Zelda's.
- B Zelda's quiz average for the academic year was higher than Yolanda's.
- C Yolanda's quiz average for the academic year was 3 points higher than Zelda's.
- D Zelda's quiz average for the academic year equaled Yolanda's.
- E If Zelda had scored 3 points higher on each quiz she took, then she would have had the same average for the academic year as Yolanda.

Solution:

Set Zelda's first-semester average to $0.$ Then Yolanda's first semester is $3,$ her second semester is $3 + 18 = 21,$ and Zelda's second semester is $21 - 3 = 18.$ Each person's yearly average is a weighted average of their two semester averages, so Yolanda's year average lies between 3 and $21,$ and Zelda's lies between 0 and $18.$ The largest possible gap Yolanda $-$ Zelda is therefore at most $21 - 0 = 21,$ so it can never be $22.$ All the other statements are achievable for suitable quiz counts.

Thus, the correct answer is **A**.

19. Each of 2023 balls is placed in one of 3 bins. Which of the following is closest to the probability that each of the bins will contain an odd number of balls?

A $\frac{2}{3}$

B $\frac{3}{10}$

C $\frac{1}{2}$

D $\frac{1}{3}$

E $\frac{1}{4}$

Solution:

Counting assignments where all three bins are odd with the parity filter gives

$$\frac{1}{8}\sum_{S\subseteq\{1,2,3\}}(-1)^{|S|}(3-2|S|)^n=\frac{3^n-3}{4}$$

for odd n. Dividing by the 3^n total assignments, the probability is $\frac{3^n-3}{4\cdot 3^n}$, which for $n=2023$ is extremely close to $\frac{1}{4}$.

Thus, the correct answer is **E**.

20. Cyrus the frog jumps 2 units in a direction, then 2 more in another direction. What is the probability that he lands less than 1 unit away from his starting position?

A $\frac{1}{6}$

B $\frac{1}{5}$

C $\frac{\sqrt{3}}{8}$

D $\frac{\arctan \frac{1}{2}}{\pi}$

E $\frac{2 \arcsin \frac{1}{4}}{\pi}$

Solution:

Take the first jump as $(2, 0)$ and the second as $(2\cos\theta, 2\sin\theta)$ with θ uniform on $[0, 2\pi)$. The landing distance satisfies $R^2 = (2 + 2\cos\theta)^2 + (2\sin\theta)^2 = 8 + 8\cos\theta$. We need $R < 1$, i.e. $\cos\theta < -\frac{7}{8}$. The measure of such angles is $2\arccos\frac{7}{8}$, so the probability is $\frac{2\arccos\frac{7}{8}}{2\pi} = \frac{\arccos\frac{7}{8}}{\pi}$. Using $\arccos(1 - 2x^2) = 2\arcsin x$ with $x = \frac{1}{4}$ gives $\arccos\frac{7}{8} = 2\arcsin\frac{1}{4}$, so the probability is $\frac{2\arcsin\frac{1}{4}}{\pi}$.

Thus, the correct answer is **E**.

21. A lampshade is made in the form of the lateral surface of the frustum of a right circular cone. The height of the frustum is $3\sqrt{3}$ inches, its top diameter is 6 inches, and its bottom diameter is 12 inches. A bug is at the bottom of the lampshade and there is a glob of honey on the top edge of the lampshade at the spot farthest from the bug. The bug wants to crawl to the honey, but it must stay on the surface of the lampshade. What is the length in inches of its shortest path to the honey?

A $6+3\pi$

B $6+6\pi$

C $6\sqrt{3}$

D $6\sqrt{5}$

E $6\sqrt{3}+\pi$

Solution:

Extend the frustum to a full cone. Since the radii are 3 and 6 with slant band $6,$ the apex is slant distance 6 from the top rim and 12 from the bottom rim. The bottom circumference 12π unrolls to a sector of radius 12 and angle $\dfrac{12\pi}{12} = \pi.$ Place the bug at $(12, 0)$ in this pattern; the honey, halfway around the base, is at radius 6 and angle $\frac{\pi}{2}.$ The straight chord between them passes within radius 6 (off the surface), so the geodesic goes tangent to the circle of radius 6 : the tangent has length $\sqrt{12^2-6^2} = 6\sqrt{3}$ and touches at angle $\frac{\pi}{3}$, after which the path follows the arc of angle $\frac{\pi}{6}$ on radius $6,$ of length $6\cdot\frac{\pi}{6} = \pi.$ The shortest path is $6\sqrt{3}+\pi.$

Thus, the correct answer is **E**.

22. A real-valued function f has the property that for all real numbers a and b,

$$f(a+b)+f(a-b)=2f(a)f(b).$$

Which one of the following cannot be the value of $f(1)$?

A 0

B 1

C -1

D 2

E -2

Solution:

Setting $a=b=0$ gives $2f(0)=2f(0)^2$, so $f(0)=0$ or $f(0)=1$. If $f(0)=0$, then setting $b=0$ forces $f\equiv 0$, giving $f(1)=0$. Otherwise $f(0)=1$, and setting $a=b$ gives $f(2a)=2f(a)^2-1\geq -1$ for every a. In particular, with $a=\frac{1}{2}$, $f(1)=2f\left(\frac{1}{2}\right)^2-1\geq -1$. So $f(1)\geq -1$, and indeed every value in $[-1,\infty)$ is attainable (e.g. $f(x)=\cos(kx)$ or $\cosh(kx)$). Hence -2 is impossible.

Thus, the correct answer is **E**.

23. When n standard six-sided dice are rolled, the product of the numbers rolled can be any of 936 possible values. What is n?

A 11

B 6

C 8

D 10

E 9

Solution:

Each die contributes an exponent vector in the primes $2, 3, 5$ (face $1 \to (0,0,0)$, $2 \to (1,0,0), 3 \to (0,1,0), 4 \to (2,0,0), 5 \to (0,0,1), 6 \to (1,1,0)$), and a product is determined by the sum of these vectors. Counting the distinct attainable sums for $n = 1, 2, 3, \ldots$ gives $6, 18, 40, 75, 126, 196, 288, 405, 550, 726, 936,$ so $n = 11.$

Thus, the correct answer is **A**.

24. Suppose that $a, b, c,$ and d are positive integers satisfying all of the following relations.

$$\begin{aligned} abcd &= 2^6 \cdot 3^9 \cdot 5^7 \\ \text{lcm}(a,b) &= 2^3 \cdot 3^2 \cdot 5^3 \\ \text{lcm}(a,c) &= 2^3 \cdot 3^3 \cdot 5^3 \\ \text{lcm}(a,d) &= 2^3 \cdot 3^3 \cdot 5^3 \\ \text{lcm}(b,c) &= 2^1 \cdot 3^3 \cdot 5^2 \\ \text{lcm}(b,d) &= 2^2 \cdot 3^3 \cdot 5^2 \\ \text{lcm}(c,d) &= 2^2 \cdot 3^3 \cdot 5^2 \end{aligned}$$

What is $\gcd(a, b, c, d)$?

A 30

B 45

C 3

D 15

E 6

Solution:

Handle each prime separately using the exponents of a, b, c, d.

Prime 2 (total 6): $\max(b, c) = 1$ forces $a = 3$; then $b + c + d = 3$ with $\max(b, d) = \max(c, d) = 2$ gives $d = 2$ and $\{b, c\} = \{0, 1\}$, so the minimum exponent is 0.

Prime 3 (total 9): $\max(a, b) = 2$ with the other lcms equal to 3 forces $c = d = 3$; then $a + b = 3$ with $\max(a, b) = 2$ gives $\{a, b\} = \{1, 2\}$, so the minimum is 1.

Prime 5 (total 7): $\max(a, b) = 3$ with $\max(b, c), \max(b, d), \max(c, d) = 2$ forces $a = 3$; then $b + c + d = 4$ with each ≤ 2 and pairwise maxima 2 gives two of them equal to 2 and one equal to 0, so the minimum is 0.

Therefore $\gcd(a, b, c, d) = 2^0 \cdot 3^1 \cdot 5^0 = 3.$

Thus, the correct answer is **C**.

25. A regular pentagon with area $\sqrt{5}+1$ is printed on paper and cut out. The five vertices of the pentagon are folded into the center of the pentagon, creating a smaller pentagon. What is the area of the new pentagon?

A $4-\sqrt{5}$

B $\sqrt{5}-1$

C $8-3\sqrt{5}$

D $\dfrac{\sqrt{5}+1}{2}$

E $\dfrac{2+\sqrt{5}}{3}$

Solution:

Let the original pentagon have circumradius R. Folding a vertex to the center creases along the perpendicular bisector of the segment from the center to that vertex, a line at distance $\frac{R}{2}$ from the center. The five creases bound a regular pentagon with apothem $\frac{R}{2}$, whereas the original has apothem $R\cos 36^\circ$. Areas scale as the square of the apothem, so the ratio is

$$\frac{(R/2)^2}{(R\cos 36^\circ)^2} = \frac{1}{4\cos^2 36^\circ}.$$

Since $\cos 36^\circ = \frac{1+\sqrt{5}}{4}$, this ratio is $\frac{4}{(1+\sqrt{5})^2} = \frac{4}{6+2\sqrt{5}} = \frac{2}{3+\sqrt{5}} = \frac{3-\sqrt{5}}{2}$. Multiplying by the original area $\sqrt{5}+1$ gives $\frac{(3-\sqrt{5})(\sqrt{5}+1)}{2} = \frac{2\sqrt{5}-2}{2} = \sqrt{5}-1.$

Thus, the correct answer is **B**.

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