2023 AMC 12A Solutions

Typeset by: LIVE by Po-Shen Loh
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1. Cities A and B are 45 miles apart. Alicia lives in A and Beth lives in B. Alicia bikes towards B at 18 miles per hour. Leaving at the same time, Beth bikes toward A at 12 miles per hour. How many miles from City A will they be when they meet?

A 20

B 24

C 25

D 26

E 27

Solution:

The gap between them closes at $18 + 12 = 30$ miles per hour, so they meet after $\frac{45}{30} = 1.5$ hours.

In that time Alicia has ridden $18 \cdot 1.5 = 27$ miles from City A.

Thus, the correct answer is **E**.

2. The weight of $\frac{1}{3}$ of a large pizza together with $3\frac{1}{2}$ cups of orange slices is the same as the weight of $\frac{3}{4}$ of a large pizza together with $\frac{1}{2}$ cup of orange slices. A cup of orange slices weighs $\frac{1}{4}$ of a pound. What is the weight, in pounds, of a large pizza?

A $1\frac{4}{5}$

B 2

C $2\frac{2}{5}$

D 3

E $3\frac{3}{5}$

Solution:

Let the pizza weigh P pounds. Then $3\frac{1}{2}$ cups weigh $\frac{7}{2} \cdot \frac{1}{4} = \frac{7}{8}$ and $\frac{1}{2}$ cup weighs $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.

The equation is

$$\frac{1}{3}P + \frac{7}{8} = \frac{3}{4}P + \frac{1}{8}.$$

Subtracting gives $\frac{7}{8} - \frac{1}{8} = \left(\frac{3}{4} - \frac{1}{3}\right) P$, so $\frac{3}{4} = \frac{5}{12}P$ and $P = \dfrac{9}{5} = 1\frac{4}{5}$.

Thus, the correct answer is **A**.

3. How many positive perfect squares less than 2023 are divisible by 5?

A 8

B 9

C 10

D 11

E 12

Solution:

A perfect square is divisible by 5 only if its root is, so the squares are $(5k)^2 = 25k^2$.

Since $44^2 = 1936 < 2023 < 2025 = 45^2$, the root can be $5, 10, \ldots, 40$, which is $k = 1$ through 8.

Thus, the correct answer is **A**.

4. How many digits are in the base-ten representation of $8^5 \cdot 5^{10} \cdot 15^5$?

- A 14
- B 15
- C 16
- D 17
- E 18

Solution:

Writing everything in primes,

$$8^5 \cdot 5^{10} \cdot 15^5 = 2^{15} \cdot 5^{10} \cdot 3^5 \cdot 5^5 = 3^5 \cdot 2^{15} \cdot 5^{15}.$$

This equals $3^5 \cdot 10^{15} = 243 \cdot 10^{15}$, which is 243 followed by 15 zeros, for a total of 18 digits.

Thus, the correct answer is **E**.

5. Janet rolls a standard 6-sided die 4 times and keeps a running total of the numbers she rolls. What is the probability that at some point, her running total will equal 3?

A $\frac{2}{9}$

B $\frac{49}{216}$

C $\frac{25}{108}$

D $\frac{17}{72}$

E $\frac{13}{54}$

Solution:

The running total is increasing, so it hits 3 exactly when one of these disjoint openings occurs: a first roll of 3; rolls $1, 2$; rolls $2, 1$; or rolls $1, 1, 1$.

Their probabilities are

$$\frac{1}{6}+\frac{1}{36}+\frac{1}{36}+\frac{1}{216}=\frac{36+6+6+1}{216}=\frac{49}{216}.$$

Thus, the correct answer is **B**.

6. Points A and B lie on the graph of $y = \log_2 x$. The midpoint of $\overline{AB}$ is $(6, 2)$. What is the positive difference between the x-coordinates of A and B?

- A $2\sqrt{11}$
- B $4\sqrt{3}$
- C 8
- D $4\sqrt{5}$
- E 9

Solution:

Let the x-coordinates be x_1 and x_2. The midpoint gives $x_1 + x_2 = 12$, and the average of the y-values gives $\log_2 x_1 + \log_2 x_2 = 4$, so $x_1 x_2 = 2^4 = 16$.

Then

$$|x_1 - x_2| = \sqrt{(x_1 + x_2)^2 - 4x_1 x_2} = \sqrt{144 - 64} = \sqrt{80} = 4\sqrt{5}.$$

Thus, the correct answer is **D**.

7. A digital display shows the current date as an 8-digit integer consisting of a 4-digit year, followed by a 2-digit month, followed by a 2-digit date within the month. For example, Arbor Day this year is displayed as $20230428.$ For how many dates in 2023 will each digit appear an even number of times in the 8-digit display for that date?

A 5

B 6

C 7

D 8

E 9

Solution:

The year contributes the digits $2, 0, 2, 3,$ so 2 appears twice while 0 and 3 each appear once. For every digit to end up with an even count, the four digits of the month and day must supply an odd number of 0's, an odd number of 3's, and an even number of every other digit.

With only four digits available, the month-day string must use exactly one $0,$ one $3,$ and a repeated pair of some digit. Checking valid months and days leaves nine dates: $01\text{-}13, 01\text{-}31, 02\text{-}23, 03\text{-}11, 03\text{-}22, 10\text{-}13, 10\text{-}31, 11\text{-}03,$ and $11\text{-}30.$

Thus, the correct answer is **E**.

8. Maureen is keeping track of the mean of her quiz scores this semester. If Maureen scores an 11 on the next quiz, her mean will increase by 1. If she scores an 11 on each of the next three quizzes, her mean will increase by 2. What is the mean of her quiz scores currently?

- A 4
- B 5
- C 6
- D 7
- E 8

Solution:

Let the current mean be m over n quizzes, so the total is $S = mn$. Adding one 11 gives

$$\frac{mn+11}{n+1} = m+1,$$

which simplifies to $m+n=10$.

Adding three 11's gives

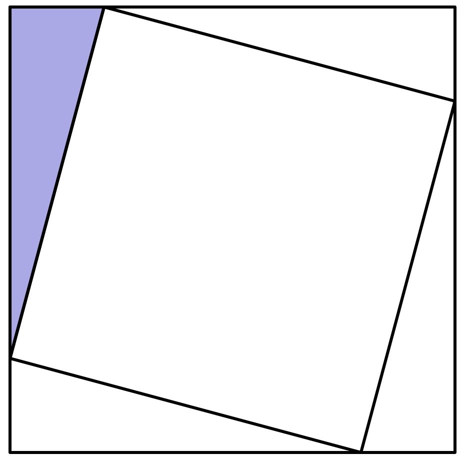
$$\frac{mn+33}{n+3} = m+2,$$

which simplifies to $3m+2n=27$.

Solving $m+n=10$ and $3m+2n=27$ gives $m=7$.

Thus, the correct answer is **D**.

9. A square of area 2 is inscribed in a square of area $3,$ creating four congruent triangles, as shown below. What is the ratio of the shorter leg to the longer leg in the shaded right triangle?



A $\frac{1}{5}$

B $\frac{1}{4}$

C $2-\sqrt{3}$

D $\sqrt{3}-\sqrt{2}$

E $\sqrt{2}-1$

Solution:

The outer square has side $\sqrt{3}$ and the inner square has side $\sqrt{2}.$ Each triangle is right, with legs p and q along an outer side, so $p+q=\sqrt{3},$ and with hypotenuse an inner side, so $p^2+q^2=2.$

Then $(p+q)^2=3$ gives $2pq=1,$ so p and q are the roots of $t^2-\sqrt{3}\,t+\frac{1}{2}=0,$ namely $\dfrac{\sqrt{3}\pm 1}{2}.$

The ratio of shorter to longer leg is

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}-1)^2}{2}=2-\sqrt{3}.$$

Thus, the correct answer is **C**.

10. Positive real numbers x and y satisfy $y^3 = x^2$ and $(y - x)^2 = 4y^2$. What is $x + y$?

- A 12
- B 18
- C 24
- D 36
- E 42

Solution:

From $(y - x)^2 = 4y^2$ we get $y - x = \pm 2y$. The choice $y - x = 2y$ gives $x = -y < 0$, impossible, so $y - x = -2y$, meaning $x = 3y$.

Substituting into $y^3 = x^2 = 9y^2$ gives $y = 9$, hence $x = 27$ and $x + y = 36$.

Thus, the correct answer is **D**.

11. What is the degree measure of the acute angle formed by lines with slopes 2 and $\frac{1}{3}$?

A 30

B 37.5

C 45

D 52.5

E 60

Solution:

The tangent of the angle between the lines is

$$\left|\frac{2-\frac{1}{3}}{1+2\cdot\frac{1}{3}}\right| = \left|\frac{\frac{5}{3}}{\frac{5}{3}}\right| = 1.$$

The acute angle with tangent 1 is 45°.

Thus, the correct answer is **C**.

12. What is the value of

$$2^3 - 1^3 + 4^3 - 3^3 + 6^3 - 5^3 + \cdots + 18^3 - 17^3?$$

A 2023

B 2679

C 2941

D 3159

E 3235

Solution:

Group into pairs $(2k)^3 - (2k-1)^3$ for $k = 1, \ldots, 9$. Expanding,

$$(2k)^3 - (2k-1)^3 = 12k^2 - 6k + 1.$$

Summing for $k = 1$ to 9, with $\sum k^2 = 285$ and $\sum k = 45$, gives

$$12 \cdot 285 - 6 \cdot 45 + 9 = 3420 - 270 + 9 = 3159.$$

Thus, the correct answer is **D**.

13. In a table tennis tournament every participant played every other participant exactly once. Although there were twice as many right-handed players as left-handed players, the number of games won by left-handed players was 40% more than the number of games won by right-handed players. (There were no ties and no ambidextrous players.) What is the total number of games played?

- A 15
- B 36
- C 45
- D 48
- E 66

Solution:

Let there be L left-handed and $2L$ right-handed players, for $3L$ players and $\binom{3L}{2}$ games total.

If right-handers win R games, left-handers win $1.4R$, so the total is $2.4R = \frac{12}{5}R$. For this to be an integer count, the total number of games must be a multiple of 12.

Testing $L = 1, 2, 3$ gives totals $3, 15, 36$; only 36 is a multiple of 12, and it is achievable (the 3 left-handers can take all 18 mixed games plus their 3 internal games for $21 = 1.4 \cdot 15$ wins).

Thus, the correct answer is **B**.

14. How many complex numbers satisfy the equation $z^5 = \overline{z}$, where $\overline{z}$ is the conjugate of the complex number z?

A 2

B 3

C 5

D 6

E 7

Solution:

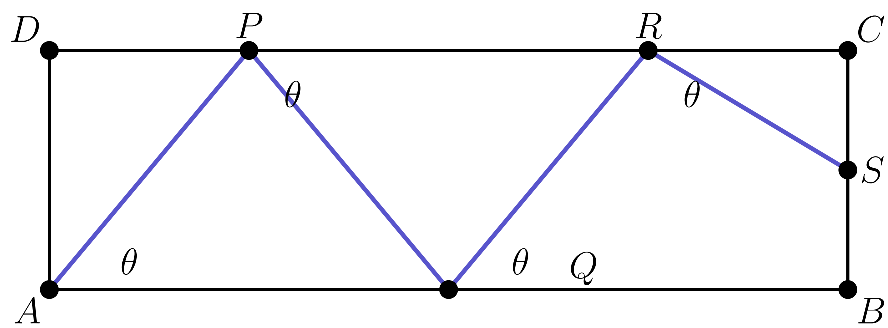
Taking magnitudes gives $|z|^5 = |z|$, so $|z| = 0$ or $|z| = 1$. The value $z = 0$ works, giving one solution.

If $|z| = 1$, multiply the equation by z to get $z^6 = z\overline{z} = |z|^2 = 1$. This has 6 distinct roots, all of modulus 1.

Altogether there are $1 + 6 = 7$ solutions.

Thus, the correct answer is **E**.

15. Usain is walking for exercise by zigzagging across a 100-meter by 30-meter rectangular field, beginning at point A and ending on the segment $\overline{BC}$. He wants to increase the distance walked by zigzagging as shown in the figure below ($APQRS$). What angle $\theta = \angle PAB = \angle QPC = \angle RQB = \cdots$ will produce a length that is 120 meters? (Do not assume the zigzag path has exactly four segments as shown; there could be more or fewer.)



A $\arccos \frac{5}{6}$

B $\arccos \frac{4}{5}$

C $\arccos \frac{3}{10}$

D $\arcsin \frac{4}{5}$

E $\arcsin \frac{5}{6}$

Solution:

Every segment of the zigzag spans the full width 30, so it has length $\dfrac{30}{\sin\theta}$ and advances $\dfrac{30}{\tan\theta}$ horizontally.

Summing over all segments, the total length is 120 and the total horizontal advance is 100. Their ratio is

$$\frac{\text{length}}{\text{horizontal}} = \frac{1/\sin\theta}{1/\tan\theta} = \frac{1}{\cos\theta} = \frac{120}{100}.$$

Therefore $\cos\theta = \dfrac{5}{6}$, so $\theta = \arccos\dfrac{5}{6}$.

Thus, the correct answer is **A**.

16. Consider the set of complex numbers z satisfying $|1+z+z^2|=4$. The maximum value of the imaginary part of z can be written in the form $\frac{\sqrt{m}}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

- A 20
- B 21
- C 22
- D 23
- E 24

Solution:

Write $z=x+yi$. Then $1+z+z^2=(1+x+x^2-y^2)+y(1+2x)i$, and the constraint is

$$(1+x+x^2-y^2)^2+y^2(1+2x)^2=16.$$

Setting the derivative of y with respect to x to zero factors as $(1+2x)\left(P+2y^2\right)=0$, where $P=1+x+x^2-y^2$. The factor $P+2y^2=0$ is impossible for real x,y, so $x=-\frac{1}{2}$.

Then $1+2x=0$, so the constraint reduces to $\left(\frac{3}{4}-y^2\right)^2=16$. Taking $\frac{3}{4}-y^2=-4$ gives $y^2=\frac{19}{4}$, so the maximum is $y=\frac{\sqrt{19}}{2}$.

Here $m=19$ and $n=2$, so $m+n=21$.

Thus, the correct answer is **B**.

17. Flora the frog starts at 0 on the number line and makes a sequence of jumps to the right. In any one jump, independent of previous jumps, Flora leaps a positive integer distance m with probability $\frac{1}{2^m}$. What is the probability that Flora will eventually land at 10?

A $\frac{5}{512}$

B $\frac{45}{1024}$

C $\frac{127}{1024}$

D $\frac{511}{1024}$

E $\frac{1}{2}$

Solution:

Let a_n be the probability that Flora ever lands exactly on n, with $a_0 = 1$. Conditioning on the first jump,

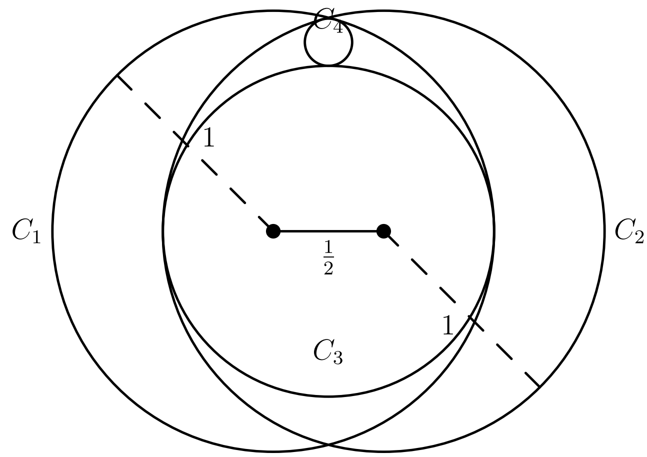
$$a_n = \sum_{k=1}^{n} \frac{1}{2^k} a_{n-k}.$$

Then $a_1 = \frac{1}{2}, a_2 = \frac{1}{2}$, and by induction $a_n = \frac{1}{2}$ for all $n \geq 1$: each new term averages the previous values, all equal to $\frac{1}{2}$.

Hence the probability of landing on 10 is $\frac{1}{2}$.

Thus, the correct answer is **E**.

18. Circle C_1 and C_2 each have radius 1, and the distance between their centers is $\frac{1}{2}$. Circle C_3 is the largest circle internally tangent to both C_1 and C_2. Circle C_4 is internally tangent to both C_1 and C_2 and externally tangent to C_3. What is the radius of C_4?



A $\dfrac{1}{14}$

B $\dfrac{1}{12}$

C $\dfrac{1}{10}$

D $\dfrac{3}{28}$

E $\dfrac{1}{9}$

Solution:

Put the centers at $O_1 = \left(-\frac{1}{4}, 0\right)$ and $O_2 = \left(\frac{1}{4}, 0\right)$. By symmetry C_3 is centered at the origin, and internal tangency to C_1 gives radius $1 - \frac{1}{4} = \frac{3}{4}$.

Let C_4 have radius r, centered at $(0, k)$ on the axis of symmetry. External tangency to C_3 gives $k = \frac{3}{4} + r$, and internal tangency to C_1 gives $\sqrt{\frac{1}{16} + k^2} = 1 - r$.

Substituting, $\frac{1}{16} + \left(\frac{3}{4} + r\right)^2 = (1 - r)^2$, which simplifies to $\frac{7}{2}r = \frac{3}{8}$, so $r = \dfrac{3}{28}$.

Thus, the correct answer is **D**.

19. What is the product of all the solutions to the equation

$$\log_{7x} 2023 \cdot \log_{289x} 2023 = \log_{2023x} 2023?$$

- A $(\log_{2023} 7 \cdot \log_{2023} 289)^2$
- B $\log_{2023} 7 \cdot \log_{2023} 289$
- C 1
- D $\log_7 2023 \cdot \log_{289} 2023$
- E $(\log_7 2023 \cdot \log_{289} 2023)^2$

Solution:

Let $a = \log_{2023} 7$ and $b = \log_{2023} 289$. Since $2023 = 7 \cdot 289$, we have $a + b = 1$. Writing $t = \log_{2023} x$, each logarithm becomes a reciprocal, and the equation turns into

$$(1 + t) = (a + t)(b + t).$$

Expanding and using $a + b = 1$, the linear terms cancel, leaving $t^2 + (ab - 1) = 0$. Its two roots satisfy $t_1 + t_2 = 0$.

The corresponding solutions multiply to $x_1 x_2 = 2023^{t_1} \cdot 2023^{t_2} = 2023^{t_1 + t_2} = 2023^0 = 1$.

Thus, the correct answer is **C**.

20. Rows $1, 2, 3, 4,$ and 5 of a triangular array of integers are shown below.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 3 & & 1 & & \\ & 1 & & 5 & & 5 & & 1 & \\ 1 & & 7 & & 11 & & 7 & & 1 \end{array}$$

Each row after the first row is formed by placing a 1 at each end of the row, and each interior entry is 1 greater than the sum of the two numbers diagonally above it in the previous row. What is the units digit of the sum of the 2023 numbers in the 2023rd row?

A 1

B 3

C 5

D 7

E 9

Solution:

Let S_n be the sum of row n. Each interior entry is 1 more than the sum of the two entries above it, and summing over the row gives the recurrence

$$S_n = 2S_{n-1} + (n-2).$$

With $S_1 = 1$, this solves to $S_n = 2^n - n$ (check: $S_5 = 32 - 5 = 27 = 1 + 7 + 11 + 7 + 1$).

So $S_{2023} = 2^{2023} - 2023$. Since powers of 2 cycle with units digits $2, 4, 8, 6$ and $2023 \equiv 3 \pmod 4$, 2^{2023} ends in 8. Then $8 - 3 = 5$ gives units digit 5.

Thus, the correct answer is **C**.

21. If A and B are vertices of a polyhedron, define the distance $d(A, B)$ to be the minimum number of edges of the polyhedron one must traverse in order to connect A and B. For example, if $\overline{AB}$ is an edge of the polyhedron, then $d(A, B) = 1$, but if $\overline{AC}$ and $\overline{CB}$ are edges and $\overline{AB}$ is not an edge, then $d(A, B) = 2$. Let Q, R, and S be randomly chosen distinct vertices of a regular icosahedron (regular polyhedron made up of 20 equilateral triangles). What is the probability that $d(Q, R) > d(R, S)$?

A $\frac{7}{22}$

B $\frac{1}{3}$

C $\frac{3}{8}$

D $\frac{5}{12}$

E $\frac{1}{2}$

Solution:

Fix R. Among the other 11 vertices of the icosahedron, 5 are at distance 1, 5 are at distance 2, and 1 (the antipode) is at distance 3.

Choosing ordered distinct Q, S, the probability that $d(Q, R) = d(R, S)$ is

$$\frac{5 \cdot 4 + 5 \cdot 4}{11 \cdot 10} = \frac{40}{110} = \frac{4}{11}.$$

By the symmetry between Q and S,

$$P(d(Q, R) > d(R, S)) = \frac{1 - \frac{4}{11}}{2} = \frac{7}{22}.$$

Thus, the correct answer is **A**.

22. Let f be the unique function defined on the positive integers such that

$$\sum_{d|n} d \cdot f\left(\frac{n}{d}\right) = 1$$

for all positive integers n, where the sum is taken over all positive divisors of n. What is $f(2023)$?

A -1536

B 96

C 108

D 116

E 144

Solution:

Setting $n = 1$ gives $f(1) = 1$. For a prime p, $n = p$ gives $f(p) + p \cdot f(1) = 1$, so $f(p) = 1 - p$. For $n = p^2$, $f(p^2) + p\, f(p) + p^2 f(1) = 1$ gives $f(p^2) = 1 - p$.

Since the defining relation is a Dirichlet convolution of multiplicative functions, f is multiplicative. With $2023 = 7 \cdot 17^2$,

$$f(2023) = f(7) \cdot f(17^2) = (1 - 7)(1 - 17) = (-6)(-16) = 96.$$

Thus, the correct answer is **B**.

23. How many ordered pairs of positive real numbers (a, b) satisfy the equation

$$(1+2a)(2+2b)(2a+b) = 32ab?$$

A 0

B 1

C 2

D 3

E an infinite number

Solution:

By AM-GM, $1+2a \geq 2\sqrt{2a}, 2+2b \geq 4\sqrt{b}$, and $2a+b \geq 2\sqrt{2ab}$. Multiplying,

$$(1+2a)(2+2b)(2a+b) \geq 16\sqrt{2a} \cdot \sqrt{b} \cdot \sqrt{2ab} = 32ab.$$

Equality requires $1 = 2a, 2 = 2b$, and $2a = b$ simultaneously. These give $a = \frac{1}{2}, b = 1$, which are consistent, so there is exactly one solution.

Thus, the correct answer is **B**.

24. Let K be the number of sequences $A_1, A_2, \ldots, A_n$ such that n is a positive integer less than or equal to 10, each A_i is a subset of $\{1, 2, 3, \ldots, 10\}$, and A_{i-1} is a subset of A_i for each i between 2 and n, inclusive. For example, $\{\}, \{5, 7\}, \{2, 5, 7\}, \{2, 5, 7\}, \{2, 5, 6, 7, 9\}$ is one such sequence, with $n = 5$. What is the remainder when K is divided by 10?

- A 1
- B 3
- C 5
- D 7
- E 9

Solution:

For a fixed length n, each element of $\{1, \ldots, 10\}$ independently either never appears or first appears in one of $A_1, \ldots, A_n$, giving $n + 1$ choices. Hence there are $(n + 1)^{10}$ chains of length n.

Summing,

$$K = \sum_{n=1}^{10} (n + 1)^{10} = \sum_{k=2}^{11} k^{10}.$$

Modulo 10, the terms $k = 2, \ldots, 11$ reduce to $4, 9, 6, 5, 6, 9, 4, 1, 0, 1$, which sum to $45 \equiv 5$.

Thus, the correct answer is **C**.

25. There is a unique sequence of integers $a_1, a_2, \cdots a_{2023}$ such that

$$\tan 2023x = \frac{a_1 \tan x + a_3 \tan^3 x + a_5 \tan^5 x + \cdots + a_{2023} \tan^{2023} x}{1 + a_2 \tan^2 x + a_4 \tan^4 x \cdots + a_{2022} \tan^{2022} x}$$

whenever $\tan 2023x$ is defined. What is a_{2023}?

- A -2023
- B -2022
- C -1
- D 1
- E 2023

Solution:

By De Moivre, $(\cos x + i \sin x)^{2023} = \cos 2023x + i \sin 2023x$. Expanding the left side and taking the ratio of imaginary to real parts gives $\tan 2023x$ as the stated rational function of $\tan x$ after dividing numerator and denominator by $\cos^{2023} x$.

The coefficient a_{2023} is the coefficient of $\tan^{2023} x$ in the numerator, which comes from the $k = 2023$ term:

$$a_{2023} = (-1)^{(2023-1)/2} \binom{2023}{2023} = (-1)^{1011} = -1.$$

Thus, the correct answer is **C**.

Problems: https://live.poshenloh.com/past-contests/amc12/2023A