2022 AMC 12B Solutions

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1. Define $x \diamond y$ to be $|x - y|$ for all real numbers x and y. What is the value of

$$(1 \diamond (2 \diamond 3)) - ((1 \diamond 2) \diamond 3)?$$

A -2

B -1

C 0

D 1

E 2

Solution:

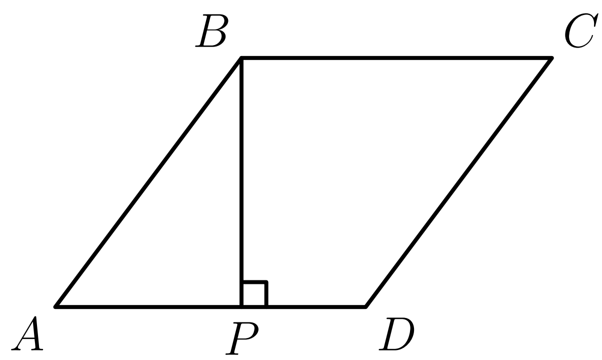
Since $2 \diamond 3 = |2 - 3| = 1$, we get $1 \diamond (2 \diamond 3) = 1 \diamond 1 = 0$.

Since $1 \diamond 2 = |1 - 2| = 1$, we get $(1 \diamond 2) \diamond 3 = 1 \diamond 3 = |1 - 3| = 2$.

The value is $0 - 2 = -2$.

Thus, the correct answer is **A**.

2. In rhombus $ABCD$, point P lies on segment $\overline{AD}$ so that $\overline{BP} \perp \overline{AD}$, $AP = 3$, and $PD = 2$. What is the area of $ABCD$? (Note: the figure is not drawn to scale.)



A $3\sqrt{5}$

B 10

C $6\sqrt{5}$

D 20

E 25

Solution:

The side length is $AD = AP + PD = 5$, so $AB = 5$. In right triangle APB,

$$BP = \sqrt{AB^2 - AP^2} = \sqrt{25 - 9} = 4.$$

Taking AD as the base and BP as the height, the area is $AD \cdot BP = 5 \cdot 4 = 20.$

Thus, the correct answer is **D**.

3. How many of the first ten numbers of the sequence $121, 11211, 1112111, \ldots$ are prime numbers?

- A 0
- B 1
- C 2
- D 3
- E 4

Solution:

The nth term consists of n ones, then a $2,$ then n ones. It factors as a repunit times a number of the form $10^n + 1:$

$$121 = 11 \cdot 11, \quad 11211 = 111 \cdot 101, \quad 1112111 = 1111 \cdot 1001,$$

and in general the nth term equals $\underbrace{1 \cdots 1}_{n+1} \cdot (10^n + 1).$

For every $n \geq 1$ both factors exceed $1,$ so every term is composite. None of the ten numbers is prime.

Thus, the correct answer is **A**.

4. For how many values of the constant k will the polynomial $x^2 + kx + 36$ have two distinct integer roots?

A 6

B 8

C 9

D 14

E 16

Solution:

If the roots are integers p and q, then $pq = 36$ and $k = -(p + q)$. Distinct roots must have the same sign, so we list factor pairs of 36 with $p \neq q$.

The positive pairs are $(1, 36), (2, 18), (3, 12), (4, 9)$, and the negative pairs are $(-1, -36), (-2, -18), (-3, -12), (-4, -9)$. The pair $(6, 6)$ is excluded since the roots must be distinct.

Each of these 8 pairs gives a different value of k.

Thus, the correct answer is **B**.

5. The point $(-1,-2)$ is rotated 270° counterclockwise about the point $(3, 1)$. What are the coordinates of its new position?

- A $(-3,-4)$
- B $(0, 5)$
- C $(2,-1)$
- D $(4, 3)$
- E $(6,-3)$

Solution:

Relative to the center $(3, 1)$, the point is at $(-1-3,\ -2-1) = (-4,-3)$.

A 270° counterclockwise rotation sends (x, y) to $(y,-x)$, so $(-4,-3)$ becomes $(-3, 4)$.

Translating back gives $(3-3,\ 1+4) = (0, 5)$.

Thus, the correct answer is **B**.

6. Consider the following 100 sets of 10 elements each:

$$\{1, 2, 3, \ldots, 10\},$$
$$\{11, 12, 13, \ldots, 20\},$$
$$\{21, 22, 23, \ldots, 30\},$$
$$\vdots$$
$$\{991, 992, 993, \ldots, 1000\}.$$

How many of these sets contain exactly two multiples of 7?

- A 40
- B 42
- C 43
- D 49
- E 50

Solution:

Among 1 to 1000 there are $\left\lfloor \frac{1000}{7} \right\rfloor = 142$ multiples of 7. Because $10 > 7$, each block of 10 consecutive integers contains one or two multiples of 7.

If x blocks contain two and the remaining $100 - x$ contain one, then $2x + (100 - x) = 142$, so $x = 42$.

Thus, the correct answer is **B**.

7. Camila writes down five positive integers. The unique mode of these integers is 2 greater than their median, and the median is 2 greater than their arithmetic mean. What is the least possible value for the mode?

A 5

B 7

C 9

D 11

E 13

Solution:

List the numbers in increasing order with median m. The mode is $m+2>m$, so it can only occur among the two largest entries; for it to be the unique mode, both of them must equal $m+2$.

The mean is $m-2$, so the total is $5(m-2)$. With the two largest equal to $m+2$ and the median m, the two smallest sum to $5(m-2)-m-2(m+2)=2m-14$.

The two smallest are distinct positive integers, so $2m-14\geq 1+2=3$, giving $m\geq 9$. With $m=9$ the list $1,3,9,11,11$ works, so the least mode is $m+2=11$.

Thus, the correct answer is **D**.

8. What is the graph of $y^4 + 1 = x^4 + 2y^2$ in the coordinate plane?

- A two intersecting parabolas
- B two nonintersecting parabolas
- C two intersecting circles
- D a circle and a hyperbola
- E a circle and two parabolas

Solution:

Rearranging, $y^4 - 2y^2 + 1 = x^4$, so $(y^2 - 1)^2 = (x^2)^2$. This factors as

$$(y^2 - 1 - x^2)(y^2 - 1 + x^2) = 0.$$

Thus either $y^2 - x^2 = 1$, which is a hyperbola, or $x^2 + y^2 = 1$, which is a circle.

Thus, the correct answer is **D**.

9. The sequence $a_0, a_1, a_2, \cdots$ is a strictly increasing arithmetic sequence of positive integers such that

$$2^{a_7} = 2^{27} \cdot a_7.$$

What is the minimum possible value of a_2?

A 8

B 12

C 16

D 17

E 22

Solution:

Dividing by 2^{27}, we need $2^{a_7 - 27} = a_7$. The only positive integer solution is $a_7 = 32$, since $2^{32-27} = 2^5 = 32$.

With common difference $d \geq 1$, we have $a_7 = a_0 + 7d = 32$ and $a_2 = a_0 + 2d = 32 - 5d$. To minimize a_2 we maximize d; since $a_0 = 32 - 7d \geq 1$, the largest choice is $d = 4$ (giving $a_0 = 4$).

Then $a_2 = 32 - 20 = 12$.

Thus, the correct answer is **B**.

10. Regular hexagon $ABCDEF$ has side length 2. Let G be the midpoint of $\overline{AB}$, and let H be the midpoint of $\overline{DE}$. What is the perimeter of $GCHF$?

- A $4\sqrt{3}$
- B 8
- C $4\sqrt{5}$
- D $4\sqrt{7}$
- E 12

Solution:

Place the hexagon with center at the origin: $A = (-1, \sqrt{3}), B = (1, \sqrt{3}), C = (2, 0), D = (1, -\sqrt{3}), E = (-1, -\sqrt{3}), F = (-2, 0)$.

Then $G = (0, \sqrt{3})$ and $H = (0, -\sqrt{3})$. By symmetry all four sides of $GCHF$ are equal, and

$$GC = \sqrt{2^2 + (\sqrt{3})^2} = \sqrt{7}.$$

The perimeter is $4\sqrt{7}$.

Thus, the correct answer is **D**.

11. Let

$$f(n) = \left(\frac{-1+i\sqrt{3}}{2}\right)^n + \left(\frac{-1-i\sqrt{3}}{2}\right)^n,$$

where $i = \sqrt{-1}$. What is $f(2022)$?

A -2

B -1

C 0

D $\sqrt{3}$

E 2

Solution:

The two bases are the primitive cube roots of unity, $\omega = e^{2\pi i/3}$ and its conjugate $\omega^2 = e^{-2\pi i/3}$. So $f(n) = \omega^n + \omega^{-n} = 2\cos\frac{2\pi n}{3}$.

Since 2022 is a multiple of $3, \omega^{2022} = 1,$ so $f(2022) = 1 + 1 = 2.$

Thus, the correct answer is **E**.

12. Kayla rolls four fair 6-sided dice. What is the probability that at least one of the numbers Kayla rolls is greater than 4 and at least two of the numbers she rolls are greater than 2?

A $\frac{2}{3}$

B $\frac{19}{27}$

C $\frac{59}{81}$

D $\frac{61}{81}$

E $\frac{7}{9}$

Solution:

Sort each die into low $\{1, 2\}$, mid $\{3, 4\}$, or high $\{5, 6\}$; each has probability $\frac{1}{3}$, so the $3^4 = 81$ category patterns are equally likely.

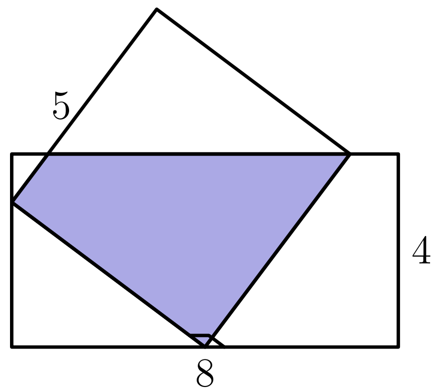
We need at least one high die (a number greater than 4) and at least two dice that are greater than 2 (mid or high). Let A be the event of at least one high and B the event of at most one low die.

There are $2^4 = 16$ patterns with no high die, 9 patterns with at most one non-low die, and 5 patterns with neither a high die nor two non-low dice. By inclusion-exclusion the count of good patterns is $81 - 16 - 9 + 5 = 61.$

The probability is $\dfrac{61}{81}$.

Thus, the correct answer is **D**.

13. The diagram below shows a rectangle with side lengths 4 and 8 and a square with side length 5. Three vertices of the square lie on three different sides of the rectangle, as shown. What is the area of the region inside both the square and the rectangle?



A $15\frac{1}{8}$

B $15\frac{3}{8}$

C $15\frac{1}{2}$

D $15\frac{5}{8}$

E $15\frac{7}{8}$

Solution:

Place the rectangle as $[0, 8] \times [0, 4]$. The tilted square, using the 3-4-5 right triangles, has vertices $(4, 0)$, $(0, 3)$, $(3, 7)$, and $(7, 4)$.

The entire square lies inside the rectangle except for the triangle poking above the top edge $y = 4$. That triangle has vertices $(0.75, 4)$, $(3, 7)$, and $(7, 4)$, with area

$$\frac{1}{2} \cdot (7 - 0.75) \cdot (7 - 4) = \frac{75}{8}.$$

The region inside both is $25 - \dfrac{75}{8} = \dfrac{125}{8} = 15\dfrac{5}{8}$.

Thus, the correct answer is **D**.

14. The graph of $y = x^2 + 2x - 15$ intersects the x-axis at points A and C and the y-axis at point B. What is $\tan(\angle ABC)$?

A $\frac{1}{7}$

B $\frac{1}{4}$

C $\frac{3}{7}$

D $\frac{1}{2}$

E $\frac{4}{7}$

Solution:

Factoring, $x^2 + 2x - 15 = (x+5)(x-3)$, so $A = (-5, 0)$ and $C = (3, 0)$, and the y-intercept is $B = (0, -15)$.

Then $\vec{BA} = (-5, 15)$ and $\vec{BC} = (3, 15)$. Using the cross and dot products,

$$\tan(\angle ABC) = \frac{|(-5)(15) - (15)(3)|}{(-5)(3) + (15)(15)} = \frac{120}{210} = \frac{4}{7}.$$

Thus, the correct answer is **E**.

15. One of the following numbers is not divisible by any prime number less than 10. Which is it?

A $2^{606} - 1$

B $2^{606} + 1$

C $2^{607} - 1$

D $2^{607} + 1$

E $2^{607} + 3^{607}$

Solution:

Every option is odd, so only the primes $3, 5, 7$ need checking.

Option A: $2^{606} \equiv 1 \pmod{3}$, so $2^{606} - 1$ is divisible by 3. Option B: $2^{606} \equiv 4 \pmod{5}$, so $2^{606} + 1$ is divisible by 5. Option D: $2^{607} \equiv 2 \pmod{3}$, so $2^{607} + 1$ is divisible by 3. Option E: modulo $5, 2^{607} + 3^{607} \equiv 3 + 2 = 5 \equiv 0.$

For $2^{607} - 1:$ it is $\equiv 1 \pmod{3}, \equiv 2 \pmod{5}$, and (since $2^3 \equiv 1 \pmod{7}$ and $607 \equiv 1 \pmod{3}$) $\equiv 1 \pmod{7}$. So it is not divisible by any prime below $10.$

Thus, the correct answer is **C**.

16. Suppose x and y are positive real numbers such that

$$x^y = 2^{64} \quad \text{and} \quad (\log_2 x)^{\log_2 y} = 2^7.$$

What is the greatest possible value of $\log_2 y$?

- A 3
- B 4
- C $3 + \sqrt{2}$
- D $4 + \sqrt{3}$
- E 7

Solution:

Let $a = \log_2 x$ and $b = \log_2 y$. Taking $\log_2$ of $x^y = 2^{64}$ gives $y \log_2 x = 64$, i.e. $a \cdot 2^b = 2^6$.

Taking $\log_2$ of the second equation gives $b \log_2 a = 7$, so $a = 2^{7/b}$. Substituting, $2^{7/b} \cdot 2^b = 2^6$, so $b + \dfrac{7}{b} = 6$, i.e. $b^2 - 6b + 7 = 0$.

Thus $b = 3 \pm \sqrt{2}$, and the greatest value of $\log_2 y$ is $3 + \sqrt{2}$.

Thus, the correct answer is **C**.

17. How many 4×4 arrays whose entries are 0s and 1s are there such that the row sums (the sum of the entries in each row) are $1, 2, 3,$ and $4,$ in some order, and the column sums (the sum of the entries in each column) are also $1, 2, 3,$ and $4,$ in some order? For example, the array

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

satisfies the condition.

A 144

B 240

C 336

D 576

E 624

Solution:

The row with sum 4 is all 1s and the column with sum 4 is all 1s. There are $4!$ ways to assign the row sums $1, 2, 3, 4$ to the four rows, and 4 choices for which column has sum $4.$

Delete that column. The remaining 4×3 array has row sums $0, 1, 2, 3$ and must have column sums $1, 2, 3.$ The all-zero and all-one rows are forced; the rows of reduced sum 1 and 2 can be placed in 6 ways to produce column sums $1, 2, 3$ in some order.

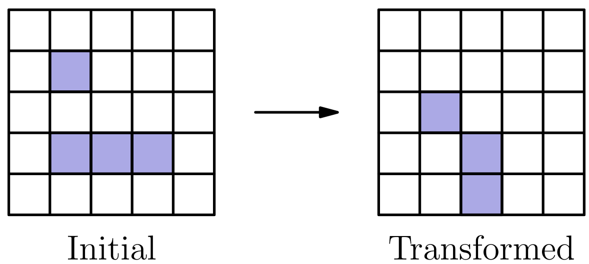
The total is $24 \cdot 4 \cdot 6 = 576.$

Thus, the correct answer is **D**.

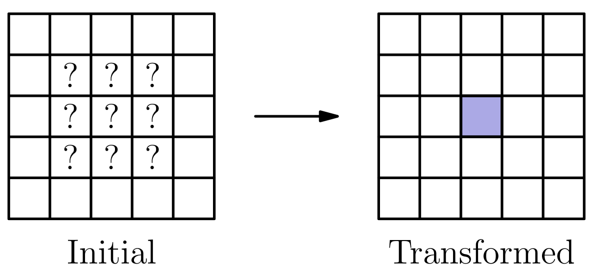
18. Each square in a 5×5 grid is either filled or empty, and has up to eight adjacent neighboring squares, where neighboring squares share either a side or a corner. The grid is transformed by the following rules:

Any filled square with two or three filled neighbors remains filled. Any empty square with exactly three filled neighbors becomes a filled square. All other squares remain empty or become empty.

A sample transformation is shown in the figure below.



Suppose the 5×5 grid has a border of empty squares surrounding a 3×3 subgrid. How many initial configurations will lead to a transformed grid consisting of a single filled square in the center after a single transformation? (Rotations and reflections of the same configuration are considered different.)



A 14

B 18

C 22

D 26

E 30

Solution:

Only the inner 3×3 squares can start filled. For the center to be filled afterward, if it began empty it needs exactly 3 filled neighbors, and if it began filled it needs 2 or $3.$

Every other square must end empty. The key restriction is that no border square may acquire exactly three filled neighbors, which rules out filling all three squares along an outer edge of the $3 \times 3.$

Enumerating the arrangements subject to these conditions, one finds every valid configuration has exactly three filled cells: there are 20 with the center initially empty and 2 with the center initially filled, for 22 in total.

Thus, the correct answer is **C**.

19. In $\triangle ABC$ medians $\overline{AD}$ and $\overline{BE}$ intersect at G and $\triangle AGE$ is equilateral. Then $\cos(C)$ can be written as $\dfrac{m\sqrt{p}}{n}$, where m and n are relatively prime positive integers and p is a positive integer not divisible by the square of any prime. What is $m+n+p$?

A 44

B 48

C 52

D 56

E 60

Solution:

Let $a=BC, b=CA, c=AB$. Since E is the midpoint of AC, $AE=\frac{b}{2}$. The centroid gives $AG=\frac{2}{3}m_a$ and $GE=\frac{1}{3}m_b$, where m_a, m_b are the medians from A and B.

Equilateral $\triangle AGE$ means $AG=GE=AE$. From $\frac{2}{3}m_a=\frac{b}{2}$ we get $m_a=\frac{3}{4}b$, which with $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ gives $2c^2-a^2=\frac{b^2}{4}$. From $\frac{1}{3}m_b=\frac{b}{2}$ we get $m_b=\frac{3}{2}b$, giving $a^2+c^2=5b^2$.

Solving, $c^2=\frac{7b^2}{4}$ and $a^2=\frac{13b^2}{4}$. Taking $b=2$ gives $a^2=13, c^2=7$, so

$$\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{13+4-7}{2\cdot\sqrt{13}\cdot 2}=\frac{5\sqrt{13}}{26}.$$

Then $m+n+p=5+26+13=44$.

Thus, the correct answer is **A**.

20. Let $P(x)$ be a polynomial with rational coefficients such that when $P(x)$ is divided by the polynomial $x^2 + x + 1$, the remainder is $x + 2$, and when $P(x)$ is divided by the polynomial $x^2 + 1$, the remainder is $2x + 1$. There is a unique polynomial of least degree with these two properties. What is the sum of the squares of the coefficients of that polynomial?

A 10

B 13

C 19

D 20

E 23

Solution:

The least-degree solution is a cubic. Write $P(x) = (x+2) + (x^2+x+1)(px+q)$, which has remainder $x + 2$ upon division by $x^2 + x + 1$.

Reducing modulo $x^2 + 1$ (so $x^2 \equiv -1$) gives remainder $(q+1)x + (2-p)$. Setting this equal to $2x + 1$ gives $q = 1$ and $p = 1$.

Then $P(x) = x^3 + 2x^2 + 3x + 3$, and the sum of the squares of the coefficients is $1 + 4 + 9 + 9 = 23$.

Thus, the correct answer is **E**.

21. Let S be the set of circles in the coordinate plane that are tangent to each of the three circles with equations $x^2 + y^2 = 4, x^2 + y^2 = 64,$ and $(x - 5)^2 + y^2 = 3.$ What is the sum of the areas of all circles in S?

A 48π

B 68π

C 96π

D 102π

E 136π

Solution:

The first two circles are concentric with radii 2 and $8.$ A circle tangent to both either has radius 3 with center at distance 5 from the origin, or radius 5 with center at distance 3 from the origin.

The third circle has center $(5, 0)$ and radius $\sqrt{3}.$ Imposing tangency with it, exactly four of the radius-3 circles and four of the radius-5 circles work (two tangency types, each giving a symmetric pair).

The sum of the areas is $4 \cdot \pi(3)^2 + 4 \cdot \pi(5)^2 = 36\pi + 100\pi = 136\pi.$

Thus, the correct answer is **E**.

22. Ant Amelia starts on the number line at 0 and crawls in the following manner. For $n = 1, 2, 3,$ Amelia chooses a time duration t_n and an increment x_n independently and uniformly at random from the interval $(0, 1)$. During the nth step of the process, Amelia moves x_n units in the positive direction, using up t_n minutes. If the total elapsed time has exceeded 1 minute during the nth step, she stops at the end of that step; otherwise, she continues with the next step, taking at most 3 steps in all. What is the probability that Amelia's position when she stops will be greater than 1?

A $\frac{1}{3}$

B $\frac{1}{2}$

C $\frac{2}{3}$

D $\frac{3}{4}$

E $\frac{5}{6}$

Solution:

Because each $t_n < 1,$ Amelia always completes at least two steps. She stops after exactly two steps when $t_1 + t_2 > 1,$ which happens with probability $\frac{1}{2}$; otherwise she takes all three steps.

The increments are independent of the times. If she takes two steps, her position is $x_1 + x_2,$ and $P(x_1 + x_2 > 1) = \frac{1}{2}$. If she takes three, her position is $x_1 + x_2 + x_3,$ and $P(x_1 + x_2 + x_3 > 1) = 1 - \frac{1}{6} = \frac{5}{6}$.

The answer is $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{5}{6} = \frac{1}{4} + \frac{5}{12} = \frac{2}{3}$.

Thus, the correct answer is **C**.

23. Let $x_0, x_1, x_2, \ldots$ be a sequence of numbers, where each x_k is either 0 or 1. For each positive integer n, define

$$S_n = \sum_{k=0}^{n-1} x_k 2^k.$$

Suppose $7S_n \equiv 1 \pmod{2^n}$ for all $n \geq 1$. What is the value of the sum

$$x_{2019} + 2x_{2020} + 4x_{2021} + 8x_{2022}?$$

A 6

B 7

C 12

D 14

E 15

Solution:

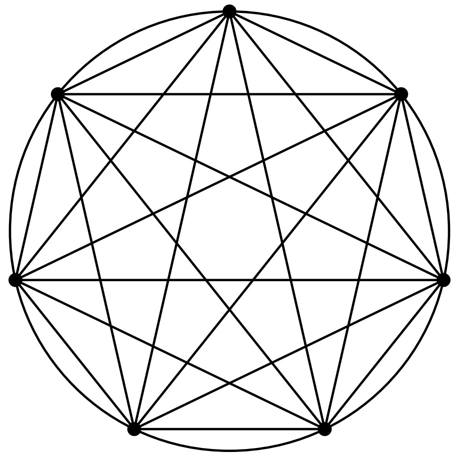
Since S_n is the integer formed by the low n bits, the condition $7S_n \equiv 1$ means $S_n \equiv 7^{-1} \pmod{2^n}$ for every n. Thus the digits x_k are the base-2 digits of $\frac{1}{7}$ as a 2-adic number.

Long division in base 2 gives digits $x_0 = x_1 = x_2 = 1$, and thereafter the block repeats with period $3:$ for $k \geq 1, x_k = 0$ exactly when $3 \mid k$, and $x_k = 1$ otherwise.

Since $3 \mid 2019$ and $3 \mid 2022$, while $2020 \equiv 1$ and $2021 \equiv 2 \pmod 3$, we get $x_{2019} = 0, x_{2020} = 1, x_{2021} = 1, x_{2022} = 0$. The sum is $0 + 2 + 4 + 0 = 6$.

Thus, the correct answer is **A**.

24. The figure below depicts a regular 7-gon inscribed in a unit circle.



What is the sum of the 4th powers of the lengths of all 21 of its edges and diagonals?

- A 49
- B 98
- C 147
- D 168
- E 196

Solution:

A chord joining two vertices d steps apart has squared length $2 - 2\cos\dfrac{2\pi d}{7}$, and there are 7 chords for each of $d = 1, 2, 3$. The required sum is

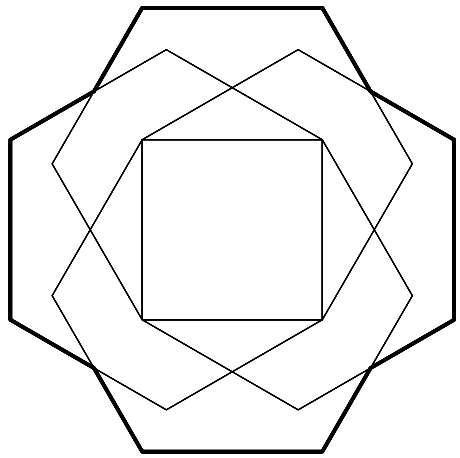
$$7\sum_{d=1}^{3}\left(2 - 2\cos\tfrac{2\pi d}{7}\right)^2.$$

Using $\sum_{d=1}^{3}\cos\frac{2\pi d}{7} = -\frac{1}{2}$ and $\sum_{d=1}^{3}\cos^2\frac{2\pi d}{7} = \frac{5}{4}$, the inner sum expands to $4\left(3 + 1 + \frac{5}{4}\right) = 21.$

Therefore the total is $7 \cdot 21 = 147.$

Thus, the correct answer is **C**.

25. Four regular hexagons surround a square with a side length 1, each one sharing an edge with the square, as shown in the figure below. The area of the resulting 12-sided outer nonconvex polygon can be written as $m\sqrt{n} + p$, where $m, n,$ and p are integers and n is not divisible by the square of any prime. What is $m + n + p$?



A -12

B -4

C 4

D 24

E 32

Solution:

Center the square at the origin with vertices $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$. Each hexagon shares one edge with the square and extends across to the opposite side; the hexagon on the bottom edge, for instance, has its far (top) edge from $\left(-\frac{1}{2}, \sqrt{3} - \frac{1}{2}\right)$ to $\left(\frac{1}{2}, \sqrt{3} - \frac{1}{2}\right)$.

The outer boundary is a 12-gon with flat edges at distance $\sqrt{3} - \frac{1}{2}$ from the center, convex vertices such as $\left(\sqrt{3} - \frac{1}{2}, \frac{1}{2}\right)$, and four reflex notches where adjacent hexagons' slanted edges meet, at $\left(\frac{5}{2} - \sqrt{3}, \frac{5}{2} - \sqrt{3}\right)$ and its symmetric images.

Applying the shoelace formula to these 12 vertices gives area $16\sqrt{3} - 23$, so $m = 16, n = 3, p = -23$, and $m + n + p = -4$.

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc12/2022B