2021 AMC 12B Fall Solutions

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1. What is the value of

$$1234 + 2341 + 3412 + 4123?$$

A $10{,}000$

B $10{,}010$

C $10{,}110$

D $11{,}000$

E $11{,}110$

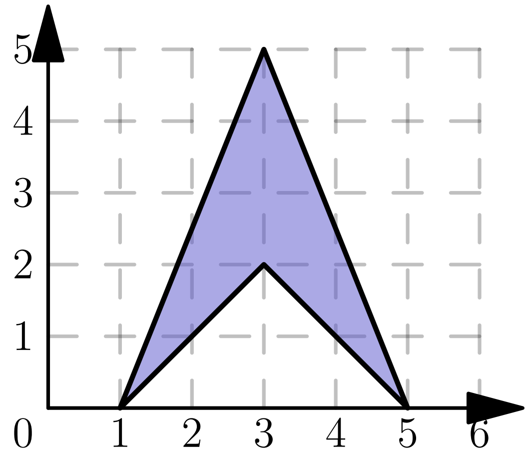
Solution:

Each of the digits $1, 2, 3, 4$ appears exactly once in the thousands, hundreds, tens, and units columns. So each column sums to $1 + 2 + 3 + 4 = 10.$

The total is therefore $10 \cdot 1111 = 11110.$

Thus, the correct answer is **E**.

2. What is the area of the shaded figure shown below?



A 4

B 6

C 8

D 10

E 12

Solution:

The outer triangle has vertices $(1,0)$, $(3,5)$, and $(5,0)$, giving base 4 and height 5, so its area is $\frac{1}{2} \cdot 4 \cdot 5 = 10$.

Removed from it is the triangle with vertices $(1,0)$, $(3,2)$, and $(5,0)$, which has base 4 and height 2, so area $\frac{1}{2} \cdot 4 \cdot 2 = 4$.

The shaded area is $10 - 4 = 6$.

Thus, the correct answer is **B**.

3. At noon on a certain day, Minneapolis is N degrees warmer than St. Louis. At $4{:}00$ the temperature in Minneapolis has fallen by 5 degrees while the temperature in St. Louis has risen by 3 degrees, at which time the temperatures in the two cities differ by 2 degrees. What is the product of all possible values of N?

- A 10
- B 30
- C 60
- D 100
- E 120

Solution:

At noon the gap is N. Minneapolis then loses 5 degrees and St. Louis gains 3, so the gap changes by 8. The new absolute difference is $|N-8|=2.$

This gives $N=10$ or $N=6,$ whose product is $60.$

Thus, the correct answer is **C**.

4. Let $n = 8^{2022}$. Which of the following is equal to

$$\frac{n}{4}?$$

A 4^{1010}

B 2^{2022}

C 8^{2018}

D 4^{3031}

E 4^{3032}

Solution:

Write $n = 8^{2022} = 2^{6066}$. Then

$$\frac{n}{4} = \frac{2^{6066}}{2^2} = 2^{6064}.$$

Since $2^{6064} = 4^{3032}$, this matches the last option.

Thus, the correct answer is **E**.

5. Call a fraction $\frac{a}{b}$, not necessarily in the simplest form, *special* if a and b are positive integers whose sum is 15. How many distinct integers can be written as the sum of two, not necessarily different, special fractions?

A 9

B 10

C 11

D 12

E 13

Solution:

The special fractions in lowest terms include the integers $2 = \frac{10}{5}, 4 = \frac{12}{3}, 14 = \frac{14}{1}$; the half-integers $\frac{1}{2}, \frac{3}{2}, \frac{13}{2}$; the quarter-integers $\frac{1}{4}$ and $\frac{11}{4}$; and others.

Two specials add to an integer only when their fractional parts cancel:

Integer pairs give $4, 6, 8, 16, 18, 28$. Half-integer pairs $\left(\frac{1}{2}, \frac{3}{2}, \frac{13}{2}\right)$ give $1, 2, 3, 7, 8, 13$. The quarter pair $\frac{1}{4} + \frac{11}{4}$ gives 3.

The distinct integers are $1, 2, 3, 4, 6, 7, 8, 13, 16, 18, 28$, a total of 11.

Thus, the correct answer is **C**.

6. The greatest prime number that is a divisor of $16{,}384$ is 2 because $16{,}384 = 2^{14}$. What is the sum of the digits of the greatest prime number that is a divisor of $16{,}383$?

A 3

B 7

C 10

D 16

E 22

Solution:

Since $16{,}383 = 2^{14} - 1 = (2^7 - 1)(2^7 + 1) = 127 \cdot 129$. Then $129 = 3 \cdot 43$, so $16{,}383 = 3 \cdot 43 \cdot 127$.

The greatest prime factor is 127, whose digit sum is $1 + 2 + 7 = 10$.

Thus, the correct answer is **C**.

7. Which of the following conditions is sufficient to guarantee that integers $x, y,$ and z satisfy the equation

$$x(x-y)+y(y-z)+z(z-x)=1?$$

A $x>y$ and $y=z$

B $x=y-1$ and $y=z-1$

C $x=z+1$ and $y=x+1$

D $x=z$ and $y-1=x$

E $x+y+z=1$

Solution:

The expression satisfies

$$2\left[x(x-y)+y(y-z)+z(z-x)\right]=(x-y)^2+(y-z)^2+(z-x)^2.$$

So the equation holds exactly when this sum of squares equals $2.$

Since the three differences sum to $0,$ this requires two of them to be ± 1 and one to be $0.$

Option D gives $z-x=0, x-y=-1,$ and $y-z=1,$ so the squares are $1+1+0=2.$ This works for all such integers.

Thus, the correct answer is **D**.

8. The product of the lengths of the two congruent sides of an obtuse isosceles triangle is equal to the product of the base and twice the triangle's height to the base. What is the measure, in degrees, of the vertex angle of this triangle?

A 105

B 120

C 135

D 150

E 165

Solution:

Let the congruent sides have length s, the base be b, and the height to the base be h. The given condition is $s^2 = 2bh$.

The area equals $\frac{1}{2}bh$ and also $\frac{1}{2}s^2 \sin\theta$, where θ is the vertex angle. So $bh = s^2 \sin\theta$.

Substituting $s^2 = 2bh$ gives $bh = 2bh \sin\theta$, so $\sin\theta = \frac{1}{2}$. Since the triangle is obtuse, $\theta = 150^\circ$.

Thus, the correct answer is **D**.

9. Triangle ABC is equilateral with side length 6. Suppose that O is the center of the inscribed circle of this triangle. What is the area of the circle passing through $A, O,$ and C?

A 9π

B 12π

C 18π

D 24π

E 27π

Solution:

For an equilateral triangle, O is also the circumcenter, so $OA = OC = \dfrac{6}{\sqrt{3}} = 2\sqrt{3}.$

The central angle $\angle AOC = 120^\circ.$

In triangle AOC, side $AC = 6$ is opposite the 120° angle, so the circumradius R' of this triangle satisfies

$$2R' = \frac{6}{\sin 120^\circ} = 4\sqrt{3},$$

giving $R' = 2\sqrt{3}.$

The area of the circle is $\pi(2\sqrt{3})^2 = 12\pi.$

Thus, the correct answer is **B**.

10. What is the sum of all possible values of t between 0 and 360 such that the triangle in the coordinate plane whose vertices are

$$(\cos 40^\circ, \sin 40^\circ),\ (\cos 60^\circ, \sin 60^\circ),\ \text{and } (\cos t^\circ, \sin t^\circ)$$

is isosceles?

A 100

B 150

C 330

D 360

E 380

Solution:

The three points lie on the unit circle at angles $40^\circ, 60^\circ$, and t°. A chord's length depends only on the angular separation of its endpoints.

If the third point is equidistant from the other two, it lies on the perpendicular bisector: $t = 50$ or $t = 230$.

If its distance to 40° equals the fixed chord (separation 20°), then $t = 20$ (since $t = 60$ is degenerate). If its distance to 60° matches, then $t = 80$ (since $t = 40$ is degenerate).

The valid values are $50, 230, 20, 80,$ summing to $380.$

Thus, the correct answer is **E**.

11. Una rolls 6 standard 6-sided dice simultaneously and calculates the product of the 6 numbers obtained. What is the probability that the product is divisible by 4?

A $\frac{3}{4}$

B $\frac{57}{64}$

C $\frac{59}{64}$

D $\frac{187}{192}$

E $\frac{63}{64}$

Solution:

The product fails to be divisible by 4 when it has at most one factor of 2. Each die is odd with probability $\frac{1}{2}$, contributes exactly one factor of 2 (a 2 or 6) with probability $\frac{1}{3}$, and two factors (a 4) with probability $\frac{1}{6}$.

All six odd: $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$. Exactly one die a 2 or 6 and the rest odd: $6 \cdot \frac{1}{3} \cdot \left(\frac{1}{2}\right)^5 = \frac{1}{16} = \frac{4}{64}$.

The complement is $\frac{1}{64} + \frac{4}{64} = \frac{5}{64}$, so the answer is $1 - \frac{5}{64} = \frac{59}{64}$.

Thus, the correct answer is **C**.

12. For n a positive integer, let $f(n)$ be the quotient obtained when the sum of all positive divisors of n is divided by n. For example,

$$f(14) = (1 + 2 + 7 + 14) \div 14 = \frac{12}{7}.$$

What is $f(768) - f(384)$?

A $\frac{1}{768}$

B $\frac{1}{192}$

C 1

D $\frac{4}{3}$

E $\frac{8}{3}$

Solution:

Since $768 = 2^8 \cdot 3$, its divisor sum is $(2^9 - 1)(1 + 3) = 511 \cdot 4 = 2044$, so $f(768) = \frac{2044}{768} = \frac{511}{192}$.

Since $384 = 2^7 \cdot 3$, its divisor sum is $(2^8 - 1)(1 + 3) = 255 \cdot 4 = 1020$, so $f(384) = \frac{1020}{384} = \frac{510}{192}$.

The difference is $\frac{511 - 510}{192} = \frac{1}{192}$.

Thus, the correct answer is **B**.

13. Let $c = \dfrac{2\pi}{11}$. What is the value of

$$\frac{\sin 3c \cdot \sin 6c \cdot \sin 9c \cdot \sin 12c \cdot \sin 15c}{\sin c \cdot \sin 2c \cdot \sin 3c \cdot \sin 4c \cdot \sin 5c}?$$

A -1

B $-\dfrac{\sqrt{11}}{5}$

C $\dfrac{\sqrt{11}}{5}$

D $\dfrac{10}{11}$

E 1

Solution:

Write each angle as $kc = \dfrac{2\pi k}{11}$. Reducing modulo 2π, $\sin 12c = \sin c$ and $\sin 15c = \sin 4c$.

So the numerator is $\sin 3c \cdot \sin 6c \cdot \sin 9c \cdot \sin c \cdot \sin 4c$. Cancelling the common factors $\sin c, \sin 3c, \sin 4c$ leaves $\dfrac{\sin 6c \cdot \sin 9c}{\sin 2c \cdot \sin 5c}$.

Now $\sin 9c = \sin(2\pi - 2c) = -\sin 2c$ and $\sin 6c = \sin(2\pi - 5c) = -\sin 5c$, so the ratio equals $\dfrac{(-\sin 5c)(-\sin 2c)}{\sin 2c \cdot \sin 5c} = 1.$

Thus, the correct answer is **E**.

14. Suppose that $P(z), Q(z)$, and $R(z)$ are polynomials with real coefficients, having degrees $2, 3$, and 6, respectively, and constant terms $1, 2$, and 3, respectively. Let N be the number of distinct complex numbers z that satisfy the equation

$$P(z) \cdot Q(z) = R(z).$$

What is the minimum possible value of N?

A 0

B 1

C 2

D 3

E 5

Solution:

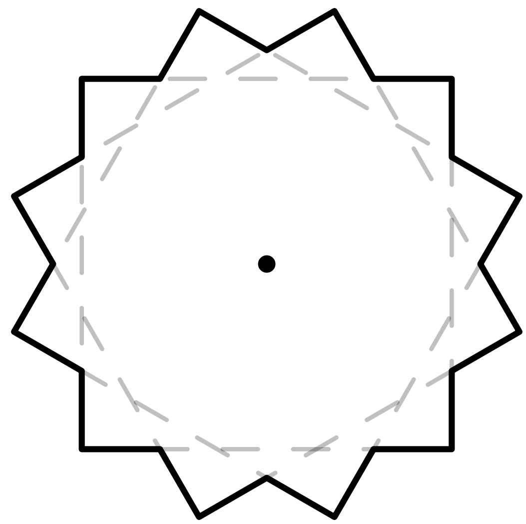
Let $D(z) = P(z)Q(z) - R(z)$. Since PQ has degree 5 and R has degree 6, the degree of D is 6. Its constant term is $1 \cdot 2 - 3 = -1 \neq 0$.

Because R is otherwise unconstrained, D can be made equal to any real degree-6 polynomial with constant term -1, for instance $-(z-1)^6$.

Such a polynomial has a single distinct root, so the minimum is $N = 1$.

Thus, the correct answer is **B**.

15. Three identical square sheets of paper each with side length 6 are stacked on top of each other. The middle sheet is rotated clockwise 30° about its center and the top sheet is rotated clockwise 60° about its center, resulting in the 24-sided polygon shown in the figure below. The area of this polygon can be expressed in the form $a - b\sqrt{c}$, where $a, b,$ and c are positive integers, and c is not divisible by the square of any prime. What is $a + b + c$?



A 75

B 93

C 96

D 129

E 147

Solution:

Because the three squares are rotated by $0^\circ, 30^\circ$, and 60°, the figure has 12-fold symmetry. Its 24 vertices alternate every 15°: outer vertices are the square corners at distance $3\sqrt{2}$ from the center, and inner vertices are edge crossings at distance $2\sqrt{3}$.

Connecting the center to all 24 vertices splits the polygon into 24 triangles, each with sides $3\sqrt{2}$ and $2\sqrt{3}$ and included angle 15°. The total area is

$$24 \cdot \tfrac{1}{2}(3\sqrt{2})(2\sqrt{3}) \sin 15^\circ = 72\sqrt{6} \cdot \frac{\sqrt{6} - \sqrt{2}}{4}.$$

This simplifies to $18\sqrt{6}(\sqrt{6} - \sqrt{2}) = 108 - 36\sqrt{3},$ so $a + b + c = 108 + 36 + 3 = 147.$

Thus, the correct answer is **E**.

16. Suppose a, b, c are positive integers such that

$$a + b + c = 23$$

and

$$\gcd(a, b) + \gcd(b, c) + \gcd(c, a) = 9.$$

What is the sum of all possible distinct values of $a^2 + b^2 + c^2$?

A 259

B 438

C 516

D 625

E 687

Solution:

The gcd sum of 9 is large, so the numbers share substantial common factors. Searching the partitions of 23 that meet the condition gives exactly two solution types.

The triple $(7, 7, 9)$ has $\gcd$ sum $7 + 1 + 1 = 9$ and $a^2 + b^2 + c^2 = 49 + 49 + 81 = 179$. The triple $(3, 5, 15)$ has $\gcd$ sum $1 + 5 + 3 = 9$ and $a^2 + b^2 + c^2 = 9 + 25 + 225 = 259$.

The sum of the distinct values is $179 + 259 = 438$.

Thus, the correct answer is **B**.

17. A bug starts at a vertex of a grid made of equilateral triangles of side length 1. At each step the bug moves in one of the 6 possible directions along the grid lines randomly and independently with equal probability. What is the probability that after 5 moves the bug never will have been more than 1 unit away from the starting position?

A $\frac{13}{108}$

B $\frac{7}{54}$

C $\frac{29}{216}$

D $\frac{4}{27}$

E $\frac{1}{16}$

Solution:

Staying within distance 1 means the bug is always at the origin or one of its 6 neighbors. From the origin, all 6 moves are allowed. From a neighbor, only 3 moves keep it in range: back to the origin, or to either of the two adjacent neighbors.

Let a_k and b_k count valid k-step paths ending at the origin and at a neighbor. Then $a_{k+1} = b_k$ and $b_{k+1} = 6a_k + 2b_k$, starting from $a_0 = 1, b_0 = 0$.

Iterating gives $b_1 = 6$, then $(a_2, b_2) = (6, 12), (a_3, b_3) = (12, 60), (a_4, b_4) = (60, 192), (a_5, b_5) = (192, 744)$. The total number of valid paths is $192 + 744 = 936$.

The probability is $\frac{936}{6^5} = \frac{936}{7776} = \frac{13}{108}$.

Thus, the correct answer is **A**.

18. Set $u_0 = \dfrac{1}{4}$, and for $k \geq 0$ let u_{k+1} be determined by the recurrence

$$u_{k+1} = 2u_k - 2u_k^2.$$

This sequence tends to a limit; call it L. What is the least value of k such that

$$|u_k - L| \leq \frac{1}{2^{1000}}?$$

A 10

B 87

C 123

D 329

E 401

Solution:

The limit satisfies $L = 2L - 2L^2$, giving $L = \frac{1}{2}$. Let $v_k = 1 - 2u_k$. Then

$$v_{k+1} = 1 - 2u_{k+1} = 1 - 4u_k + 4u_k^2 = (1 - 2u_k)^2 = v_k^2.$$

Since $v_0 = 1 - 2 \cdot \frac{1}{4} = \frac{1}{2}$, we get $v_k = \left(\frac{1}{2}\right)^{2^k}$, so $|u_k - L| = \dfrac{|v_k|}{2} = 2^{-2^k-1}$.

We need $2^k + 1 \geq 1000$, i.e. $2^k \geq 999$. The least such k is 10, since $2^{10} = 1024$.

Thus, the correct answer is **A**.

19. Regular polygons with $5, 6, 7,$ and 8 sides are inscribed in the same circle. No two of the polygons share a vertex, and no three of their sides intersect at a common point. At how many points inside the circle do two of their sides intersect?

A 52

B 56

C 60

D 64

E 68

Solution:

For two convex polygons inscribed in the same circle with no shared vertices, each side of the smaller polygon crosses the larger polygon's boundary exactly twice, so they meet at $2\min(m, n)$ points.

Summing over all pairs: $(5, 6), (5, 7), (5, 8)$ give 10 each; $(6, 7), (6, 8)$ give 12 each; $(7, 8)$ gives $14.$

The total is $3 \cdot 10 + 2 \cdot 12 + 14 = 68.$

Thus, the correct answer is **E**.

20. A cube is constructed from 4 white unit cubes and 4 blue unit cubes. How many different ways are there to construct the $2 \times 2 \times 2$ cube using these smaller cubes? (Two constructions are considered the same if one can be rotated to match the other.)

- A 7
- B 8
- C 9
- D 10
- E 11

Solution:

By Burnside's lemma, the count is the average number of 4-blue colorings fixed by each of the 24 rotations acting on the 8 cubies.

The identity fixes $\binom{8}{4} = 70.$ The 6 face quarter-turns fix 2 each $(12).$ The 3 face half-turns fix 6 each $(18).$ The 8 vertex rotations fix 4 each $(32).$ The 6 edge half-turns fix 6 each $(36).$

The total is $70 + 12 + 18 + 32 + 36 = 168$, and $\dfrac{168}{24} = 7.$

Thus, the correct answer is **A**.

21. For real numbers x, let

$$P(x) = 1 + \cos(x) + i\sin(x) - \cos(2x) - i\sin(2x) + \cos(3x) + i\sin(3x)$$

where $i = \sqrt{-1}$. For how many values of x with $0 \leq x < 2\pi$ does

$$P(x) = 0?$$

A 0

B 1

C 2

D 3

E 4

Solution:

Group by Euler's formula: $P(x) = 1 + e^{ix} - e^{2ix} + e^{3ix}$. The imaginary part is $\sin x - \sin 2x + \sin 3x = (\sin x + \sin 3x) - \sin 2x = \sin 2x(2\cos x - 1)$.

This vanishes when $\sin 2x = 0$ (so $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$) or $\cos x = \frac{1}{2}$ (so $x = \frac{\pi}{3}, \frac{5\pi}{3}$).

Checking the real part $1 + \cos x - \cos 2x + \cos 3x$ at each of these values gives ± 2 or 1, never 0. So no x makes $P(x) = 0$.

Thus, the correct answer is **A**.

22. Right triangle ABC has side lengths $BC = 6$, $AC = 8$, and $AB = 10$. A circle centered at O is tangent to line BC at B and passes through A. A circle centered at P is tangent to line AC at A and passes through B. What is OP?

A $\frac{23}{8}$

B $\frac{29}{10}$

C $\frac{35}{12}$

D $\frac{73}{25}$

E 3

Solution:

Place $C = (0, 0)$, $B = (6, 0)$, and $A = (0, 8)$, so the right angle is at C.

Circle O is tangent to line BC (the x-axis) at B, so $O = (6, k)$. Setting $OA = OB$ gives $36 + (k-8)^2 = k^2$, so $k = \frac{25}{4}$ and $O = \left(6, \frac{25}{4}\right)$.

Circle P is tangent to line AC (the y-axis) at A, so $P = (h, 8)$. Setting $PB = PA$ gives $(h-6)^2 + 64 = h^2$, so $h = \frac{25}{3}$ and $P = \left(\frac{25}{3}, 8\right)$.

Then $OP = \sqrt{\left(\frac{7}{3}\right)^2 + \left(\frac{7}{4}\right)^2} = 7\sqrt{\frac{25}{144}} = \frac{35}{12}$.

Thus, the correct answer is **C**.

23. What is the average number of pairs of consecutive integers in a randomly selected subset of 5 distinct integers chosen from the set $\{1, 2, 3, \ldots, 30\}$? (For example the set $\{1, 17, 18, 19, 30\}$ has 2 pairs of consecutive integers.)

A $\frac{2}{3}$

B $\frac{29}{36}$

C $\frac{5}{6}$

D $\frac{29}{30}$

E 1

Solution:

For each of the 29 adjacent pairs $(i, i+1)$, let an indicator be 1 if both are in the subset. The probability of this is $\frac{5}{30} \cdot \frac{4}{29} = \frac{2}{87}$.

By linearity of expectation, the expected number of consecutive pairs is $29 \cdot \frac{2}{87} = \frac{2}{3}$.

Thus, the correct answer is **A**.

24. Triangle ABC has side lengths $AB = 11, BC = 24,$ and $CA = 20.$ The bisector of $\angle BAC$ intersects $\overline{BC}$ in point $D,$ and intersects the circumcircle of $\triangle ABC$ in point $E \neq A.$ The circumcircle of $\triangle BED$ intersects the line AB in points B and $F \neq B.$ What is CF?

- A 28
- B $20\sqrt{2}$
- C 30
- D 32
- E $20\sqrt{3}$

Solution:

Points A, D, E are collinear on the bisector, and A, B, F are collinear on line $AB.$ The power of A with respect to the circle through B, E, D gives $AB \cdot AF = AD \cdot AE.$

Since $\angle BAE = \angle DAC$ and $\angle AEB = \angle ACB$ (subtending AB), triangles ABE and ADC are similar, so $AD \cdot AE = AB \cdot AC.$ Therefore $AF = AC = 20.$

Place $A = (0, 0), B = (11, 0).$ From $CA = 20, CB = 24,$ point $C = \left(-\frac{5}{2}, \frac{\sqrt{1575}}{2}\right).$ Point F lies on ray AB with $AF = 20,$ so $F = (20, 0).$

Then $CF^2 = \left(20 + \frac{5}{2}\right)^2 + \frac{1575}{4} = \frac{2025}{4} + \frac{1575}{4} = 900,$ so $CF = 30.$

Thus, the correct answer is **C**.

25. For n a positive integer, let $R(n)$ be the sum of the remainders when n is divided by 2, $3, 4, 5, 6, 7, 8, 9,$ and 10. For example,

$$R(15) = 1 + 0 + 3 + 0 + 3 + 1 + 7 + 6 + 5 = 26.$$

How many two-digit positive integers n satisfy

$$R(n) = R(n + 1)?$$

- A 0
- B 1
- C 2
- D 3
- E 4

Solution:

Going from n to $n+1$, each remainder $n \bmod m$ increases by 1 unless $m \mid n+1$, in which case it drops from $m-1$ to 0. So

$$R(n+1) - R(n) = 9 - \sum_{\substack{2 \le m \le 10 \\ m \mid n+1}} m.$$

We need those divisors to sum to 9. If $n+1$ is divisible by $3, 4, 5, 6, 8, 9,$ or 10, it picks up additional small divisors that push the sum past 9, so the only workable case is $n + 1$ divisible by 2 and 7 but no other value in $\{2, \ldots, 10\}$, giving $2 + 7 = 9$.

Among the two-digit n, this means $n + 1 = 14$ or $n + 1 = 98$, so $n = 13$ or $n = 97$. That is 2 values.

Thus, the correct answer is **C**.

Problems: https://live.poshenloh.com/past-contests/amc12/2021D