2021 AMC 12B Spring Solutions

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1. How many integer values of x satisfy $|x| < 3\pi$?

- A 9
- B 10
- C 18
- D 19
- E 20

Solution:

Since $3\pi \approx 9.42,$ the inequality $|x| < 3\pi$ means $-9.42 < x < 9.42.$

The integers in this range run from -9 to $9,$ giving 19 values.

Thus, the correct answer is **D**.

2. At a math contest, 57 students are wearing blue shirts, and another 75 students are wearing yellow shirts. The 132 students are assigned into 66 pairs. In exactly 23 of these pairs, both students are wearing blue shirts. In how many pairs are both students wearing yellow shirts?

- A 23
- **B** 32
- C 37
- D 41
- E 64

Solution:

The 23 all-blue pairs account for 46 blue students, leaving $57 - 46 = 11$ blue students.

Each of those 11 blue students must be paired with a yellow student, so there are 11 mixed pairs, using 11 yellow students.

The remaining $75 - 11 = 64$ yellow students form $64 \div 2 = 32$ all-yellow pairs.

Thus, the correct answer is **B**.

3. Suppose

$$2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+x}}}=\frac{144}{53}.$$

What is the value of x?

A $\frac{3}{4}$

B $\frac{7}{8}$

C $\frac{14}{15}$

D $\frac{37}{38}$

E $\frac{52}{53}$

Solution:

Working from the outside in, $\frac{144}{53}-2=\frac{38}{53}$, so the inner fraction equals $\frac{38}{53}$.

Its reciprocal gives $1+\frac{1}{2+\frac{2}{3+x}}=\frac{53}{38}$, so $\frac{1}{2+\frac{2}{3+x}}=\frac{15}{38}$.

Then $2+\frac{2}{3+x}=\frac{38}{15}$, so $\frac{2}{3+x}=\frac{8}{15}$, giving $3+x=\frac{15}{4}$.

Therefore $x=\frac{15}{4}-3=\frac{3}{4}$.

Thus, the correct answer is **A**.

4. Ms. Blackwell gives an exam to two classes. The mean of the scores of the students in the morning class is 84, and the afternoon class's mean score is 70. The ratio of the number of students in the morning class to the number of students in the afternoon class is $\frac{3}{4}$. What is the mean of the scores of all the students?

A 74

B 75

C 76

D 77

E 78

Solution:

Suppose there are 3 students in the morning class and 4 in the afternoon class.

The total of all scores is $3 \cdot 84 + 4 \cdot 70 = 252 + 280 = 532.$

The overall mean is $\frac{532}{7} = 76.$

Thus, the correct answer is **C**.

5. The point $P(a, b)$ in the xy-plane is first rotated counterclockwise by 90° around the point $(1, 5)$ and then reflected about the line $y = -x$. The image of P after these two transformations is at $(-6, 3)$. What is $b - a$?

A 1

B 3

C 5

D 7

E 9

Solution:

A 90° counterclockwise rotation about $(1, 5)$ sends (a, b) to $(1 - (b - 5),\ 5 + (a - 1)) = (6 - b,\ 4 + a)$.

Reflecting that about $y = -x$ (which maps (x, y) to $(-y, -x)$) gives $(-(4 + a),\ -(6 - b)) = (-4 - a,\ b - 6)$.

Setting this equal to $(-6, 3)$ gives $-4 - a = -6$ and $b - 6 = 3$, so $a = 2$ and $b = 9$.

Therefore $b - a = 9 - 2 = 7$.

Thus, the correct answer is **D**.

6. An inverted cone with base radius 12 cm and height 18 cm is full of water. The water is poured into a tall cylinder whose horizontal base has a radius of 24 cm. What is the height in centimeters of the water in the cylinder?

A 1.5

B 3

C 4

D 4.5

E 6

Solution:

The cone holds $\frac{1}{3}\pi(12)^2(18) = 864\pi$ cubic centimeters of water.

Poured into the cylinder, this fills to height h where $\pi(24)^2h = 864\pi$.

Then $576h = 864$, so $h = 1.5$.

Thus, the correct answer is **A**.

7. Let $N = 34 \cdot 34 \cdot 63 \cdot 270.$ What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

- A $1 : 16$
- B $1 : 15$
- C $1 : 14$
- D $1 : 8$
- E $1 : 3$

Solution:

Factoring, $34 = 2 \cdot 17, 63 = 3^2 \cdot 7,$ and $270 = 2 \cdot 3^3 \cdot 5,$ so $N = 2^3 \cdot 3^5 \cdot 5 \cdot 7 \cdot 17^2.$

Let M be the odd part $3^5 \cdot 5 \cdot 7 \cdot 17^2.$ The sum of all divisors is $(1 + 2 + 4 + 8)\,\sigma(M) = 15\,\sigma(M).$

The odd divisors sum to $\sigma(M),$ so the even divisors sum to $15\,\sigma(M) - \sigma(M) = 14\,\sigma(M).$

The ratio is $\sigma(M) : 14\,\sigma(M) = 1 : 14.$

Thus, the correct answer is **C**.

8. Three equally spaced parallel lines intersect a circle, creating three chords of lengths $38, 38,$ and $34.$ What is the distance between two adjacent parallel lines?

A $5\frac{1}{2}$

B 6

C $6\frac{1}{2}$

D 7

E $7\frac{1}{2}$

Solution:

Place the center at height $0.$ Two equal chords lie at equal distances from the center, so the three equally spaced lines are at heights $-\frac{d}{2}, \frac{d}{2}, \frac{3d}{2}$, with the two 38-chords at $\pm\frac{d}{2}$ and the 34-chord at $\frac{3d}{2}.$

Half-chord relations give $r^2 - \left(\frac{d}{2}\right)^2 = 19^2$ and $r^2 - \left(\frac{3d}{2}\right)^2 = 17^2.$

Subtracting, $2d^2 = 19^2 - 17^2 = 72,$ so $d^2 = 36$ and $d = 6.$

Thus, the correct answer is **B**.

9. What is the value of

$$\frac{\log_2 80}{\log_{40} 2} - \frac{\log_2 160}{\log_{20} 2}?$$

A 0

B 1

C $\frac{5}{4}$

D 2

E $\log_2 5$

Solution:

Using $\frac{1}{\log_{40} 2} = \log_2 40$ and $\frac{1}{\log_{20} 2} = \log_2 20$, the expression becomes $(\log_2 80)(\log_2 40) - (\log_2 160)(\log_2 20)$.

Let $t = \log_2 5$. Then $\log_2 80 = 4 + t, \log_2 40 = 3 + t, \log_2 160 = 5 + t, \log_2 20 = 2 + t$.

The value is $(4 + t)(3 + t) - (5 + t)(2 + t) = (12 + 7t + t^2) - (10 + 7t + t^2) = 2$.

Thus, the correct answer is **D**.

10. Two distinct numbers are selected from the set $\{1, 2, 3, 4, \ldots, 36, 37\}$ so that the sum of the remaining 35 numbers is the product of these two numbers. What is the difference of these two numbers?

- A 5
- B 7
- C 8
- D 9
- E 10

Solution:

The sum $1 + 2 + \cdots + 37 = 703$. If the chosen numbers are a and b, then $703 - a - b = ab$.

So $ab + a + b = 703$, and adding 1 gives $(a+1)(b+1) = 704 = 2^6 \cdot 11$.

We need factors $a+1, b+1$ between 2 and 38. The pair $22 \cdot 32 = 704$ works, giving $a = 21, b = 31$.

Their difference is $31 - 21 = 10$.

Thus, the correct answer is **E**.

11. Triangle ABC has $AB = 13$, $BC = 14$, and $AC = 15$. Let P be the point on $\overline{AC}$ such that $PC = 10$. There are exactly two points D and E on line BP such that quadrilaterals $ABCD$ and $ABCE$ are trapezoids. What is the distance DE?

A $\frac{42}{5}$

B $6\sqrt{2}$

C $\frac{84}{5}$

D $12\sqrt{2}$

E 18

Solution:

Place $A = (0,0)$ and $C = (15,0)$. Then $B = \left(\frac{33}{5}, \frac{56}{5}\right)$, and since $PC = 10$, $P = (5,0)$. Line BP has slope 7, so it is $y = 7(x-5)$.

For $ABCD$ to be a trapezoid with D on line BP, take $CD \parallel AB$. The line through C parallel to AB meets line BP at $(1.8, -22.4)$.

For $ABCE$ with E on line BP, take $AE \parallel BC$. The line through A parallel to BC meets line BP at $(4.2, -5.6)$.

The distance is $\sqrt{(4.2-1.8)^2 + (-5.6+22.4)^2} = \sqrt{2.4^2 + 16.8^2} = \sqrt{288} = 12\sqrt{2}$.

Thus, the correct answer is **D**.

12. Suppose that S is a finite set of positive integers. If the greatest integer in S is removed from S, then the average value (arithmetic mean) of the integers remaining is 32. If the least integer in S is also removed, then the average value of the integers remaining is 35. If the greatest integer is then returned to the set, the average value of the integers rises to 40. The greatest integer in the original set S is 72 greater than the least integer in S. What is the average value of all the integers in the set S?

A 36.2

B 36.4

C 36.6

D 36.8

E 37

Solution:

Let $n = |S|$, let T be the total, M the greatest, and L the least. Then $\dfrac{T-M}{n-1} = 32$, $\dfrac{T-M-L}{n-2} = 35$, and $\dfrac{T-L}{n-1} = 40$.

Subtracting the first from the third: $\dfrac{M-L}{n-1} = 8$. Since $M - L = 72$, we get $n - 1 = 9$, so $n = 10$.

Then $T - M = 288$ and $T - L = 360$. The middle equation gives $T - M - L = 35 \cdot 8 = 280$, so $L = 288 - 280 = 8$ and $M = 80$.

Thus $T = 288 + 80 = 368$, and the average is $\dfrac{368}{10} = 36.8$.

Thus, the correct answer is **D**.

13. How many values of θ in the interval $0 < \theta \leq 2\pi$ satisfy

$$1 - 3\sin\theta + 5\cos 3\theta = 0?$$

A 2

B 4

C 5

D 6

E 8

Solution:

Let $f(\theta) = 1 - 3\sin\theta + 5\cos 3\theta$. The fast term $5\cos 3\theta$ completes three oscillations while $1 - 3\sin\theta$ stays between -2 and 4.

Sampling f at $0, 30^\circ, 60^\circ, \ldots, 360^\circ$, the values are $+, -, -, -, +, -, -, +, +, +, -, +, +$, which shows six sign changes, hence six roots.

Each sign change corresponds to exactly one solution, so there are 6 values of θ.

Thus, the correct answer is **D**.

14. Let $ABCD$ be a rectangle and let $\overline{DM}$ be a segment perpendicular to the plane of $ABCD$. Suppose that $\overline{DM}$ has integer length, and the lengths of $\overline{MA}, \overline{MC}$, and $\overline{MB}$ are consecutive odd positive integers (in this order). What is the volume of pyramid $MABCD$?

A $24\sqrt{5}$

B 60

C $28\sqrt{5}$

D 66

E $8\sqrt{70}$

Solution:

Place D at the origin with A, C along the rectangle's edges and M directly above D. Then $MA^2 = AD^2 + DM^2, MC^2 = CD^2 + DM^2$, and $MB^2 = AD^2 + CD^2 + DM^2$.

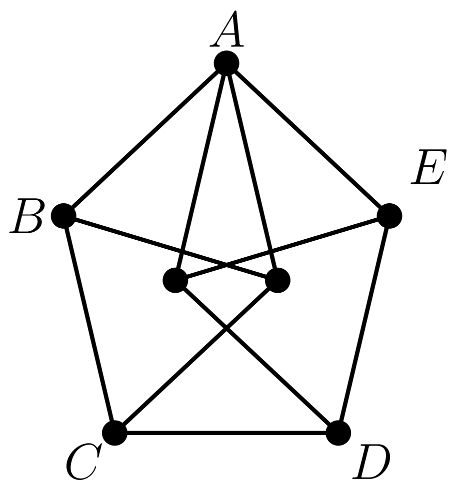
Thus $MB^2 = MA^2 + MC^2 - DM^2$. Writing $MA, MC, MB = k, k+2, k+4$, we get $DM^2 = k^2 + (k+2)^2 - (k+4)^2 = k^2 - 4k - 12$.

This is a positive perfect square only for $k = 7$, giving $DM^2 = 9$, so $DM = 3$, $MA = 7, MC = 9$. Then $AD^2 = 49 - 9 = 40$ and $CD^2 = 81 - 9 = 72$.

The base area is $AD \cdot CD = \sqrt{40} \cdot \sqrt{72} = \sqrt{2880} = 24\sqrt{5}$, and the volume is $\frac{1}{3} \cdot 24\sqrt{5} \cdot 3 = 24\sqrt{5}$.

Thus, the correct answer is **A**.

15. The figure is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written as $\sqrt{m} + \sqrt{n}$, where m and n are positive integers. What is $m + n$?



A 20

B 21

C 22

D 23

E 24

Solution:

The eleven equal segments form two rhombi (each two equilateral triangles of side 2) sharing the apex A, with C and D joined by a final segment. The figure is symmetric about the vertical line through A.

Placing $A = (0, 0)$ at the top, the two bottom vertices come out to $C = (-1, -\sqrt{11})$ and $D = (1, -\sqrt{11})$, with B and E the outer corners at height $-\frac{\sqrt{11}}{2} + \frac{1}{2\sqrt{3}}$.

Applying the shoelace formula to pentagon $ABCDE$ gives area $\sqrt{11} + 2\sqrt{3} = \sqrt{11} + \sqrt{12}$.

So $m + n = 11 + 12 = 23$.

Thus, the correct answer is **D**.

16. Let $g(x)$ be a polynomial with leading coefficient 1, whose three roots are the reciprocals of the three roots of $f(x) = x^3 + ax^2 + bx + c$, where $1 < a < b < c$. What is $g(1)$ in terms of $a, b,$ and c?

A $\dfrac{1+a+b+c}{c}$

B $1+a+b+c$

C $\dfrac{1+a+b+c}{c^2}$

D $\dfrac{a+b+c}{c^2}$

E $\dfrac{1+a+b+c}{a+b+c}$

Solution:

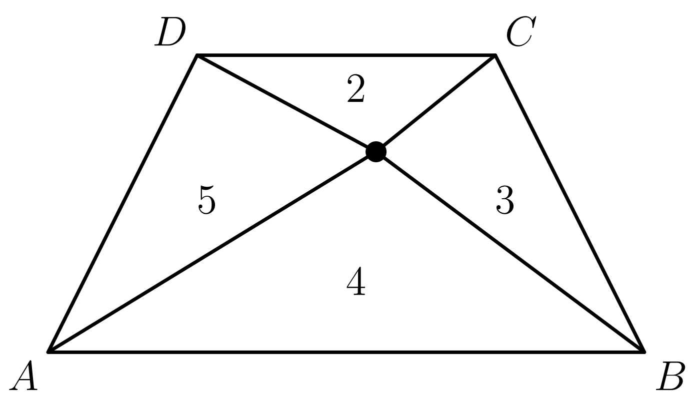
Let f have roots r, s, t. Since g is monic with roots $\frac{1}{r}, \frac{1}{s}, \frac{1}{t}$, $g(1) = \left(1-\frac{1}{r}\right)\left(1-\frac{1}{s}\right)\left(1-\frac{1}{t}\right) = \dfrac{(r-1)(s-1)(t-1)}{rst}$.

Now $f(1) = (1-r)(1-s)(1-t) = 1+a+b+c$, so $(r-1)(s-1)(t-1) = -(1+a+b+c)$. Also $rst = -c$.

Therefore $g(1) = \dfrac{-(1+a+b+c)}{-c} = \dfrac{1+a+b+c}{c}$.

Thus, the correct answer is **A**.

17. Let $ABCD$ be an isosceles trapezoid having parallel bases $\overline{AB}$ and $\overline{CD}$ with $AB > CD$. Line segments from a point inside $ABCD$ to the vertices divide the trapezoid into four triangles whose areas are $2, 3, 4,$ and 5 starting with the triangle with base $\overline{CD}$ and moving clockwise as shown in the diagram below. What is the ratio $\frac{AB}{CD}$?



A 3

B $2 + \sqrt{2}$

C $1 + \sqrt{6}$

D $2\sqrt{3}$

E $3\sqrt{2}$

Solution:

Let $AB = a, CD = b$, and let the interior point be at heights h_a from AB and h_b from CD. The base triangles give $\frac{1}{2}ah_a = 4$ and $\frac{1}{2}bh_b = 2$, so $ah_a = 8$ and $bh_b = 4$.

The total area is $2 + 3 + 4 + 5 = 14 = \frac{1}{2}(a + b)(h_a + h_b)$, so $(a + b)(h_a + h_b) = 28$. Expanding, $ah_a + bh_b + ah_b + bh_a = 28$, giving $ah_b + bh_a = 16$.

Let $u = ah_b$ and $v = bh_a$. Then $u + v = 16$ and $uv = (ah_a)(bh_b) = 32$, so $u, v = 8 \pm 4\sqrt{2}$.

Finally $\frac{AB}{CD} = \frac{a}{b} = \frac{ah_b}{bh_b} = \frac{u}{4} = \frac{8 + 4\sqrt{2}}{4} = 2 + \sqrt{2}$.

Thus, the correct answer is **B**.

18. Let z be a complex number satisfying $12|z|^2 = 2|z+2|^2 + |z^2+1|^2 + 31$. What is the value of $z + \frac{6}{z}$?

A -2

B -1

C $\frac{1}{2}$

D 1

E 4

Solution:

Let $p = |z|^2 = z\bar{z}$ and $s = z + \bar{z}$. Then $|z+2|^2 = p + 2s + 4$, and $|z^2+1|^2 = p^2 + (z^2 + \bar{z}^2) + 1 = p^2 + (s^2 - 2p) + 1$.

Substituting, $12p = 2(p + 2s + 4) + p^2 + s^2 - 2p + 1 + 31$, which simplifies to $p^2 - 12p + s^2 + 4s + 40 = 0$.

Completing the square gives $(p-6)^2 + (s+2)^2 = 0$, so $p = 6$ and $s = -2$.

Then $z + \frac{6}{z} = z + \frac{6\bar{z}}{|z|^2} = z + \bar{z} = -2$.

Thus, the correct answer is **A**.

19. Two fair dice, each with at least 6 faces are rolled. On each face of each die is printed a distinct integer from 1 to the number of faces on that die, inclusive. The probability of rolling a sum of 7 is $\dfrac{3}{4}$ of the probability of rolling a sum of $10,$ and the probability of rolling a sum of 12 is $\dfrac{1}{12}$. What is the least possible number of faces on the two dice combined?

A 16

B 17

C 18

D 19

E 20

Solution:

Let the dice have $a \leq b$ faces. Since both have at least 6 faces, a sum of 7 occurs in exactly 6 ways, so a sum of 10 occurs in $6 \div \frac{3}{4} = 8$ ways.

The number of ways to roll 10 is $\min(a, 9) - \max(1, 10 - b) + 1 = 8$. A sum of 12 has probability $\frac{1}{12}$, so it occurs in $\frac{ab}{12}$ ways.

Trying $a = 8, b = 9$: sum 10 has $8 - 1 + 1 = 8$ ways, and sum 12 has $6 = \frac{72}{12}$ ways. Both conditions hold, giving $a + b = 17$.

Checking all smaller totals $a + b = 16$ fails, so 17 is minimal.

Thus, the correct answer is **B**.

20. Let $Q(z)$ and $R(z)$ be the unique polynomials such that

$$z^{2021}+1=(z^2+z+1)Q(z)+R(z)$$

and the degree of R is less than 2. What is $R(z)$?

- A $-z$
- B -1
- C 2021
- D $z+1$
- E $2z+1$

Solution:

Since $z^3 \equiv 1 \pmod{z^2+z+1}$ and $2021 = 3 \cdot 673 + 2$, we have $z^{2021} \equiv z^2$.

So $z^{2021}+1 \equiv z^2+1$. Reducing further with $z^2 \equiv -z-1$, this is $-z-1+1 = -z$.

Therefore $R(z) = -z$.

Thus, the correct answer is **A**.

21. Let S be the sum of all positive real numbers x for which

$$x^{2^{\sqrt{2}}} = \sqrt{2}^{2^x}.$$

Which of the following statements is true?

A $S < \sqrt{2}$

B $S = \sqrt{2}$

C $\sqrt{2} < S < 2$

D $2 \le S < 6$

E $S \ge 6$

Solution:

Taking $\log_2$, the equation becomes $2^{\sqrt{2}} \log_2 x = 2^{x-1}$. Substituting $x = \sqrt{2}$ gives $2^{\sqrt{2}} \cdot \frac{1}{2} = 2^{\sqrt{2}-1}$, which holds, so $x = \sqrt{2}$ is a solution.

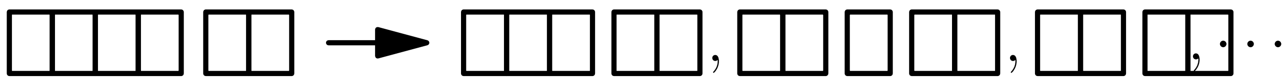
Let $f(x) = 2^{x-1} - 2^{\sqrt{2}} \log_2 x$. Then $f(1) > 0$, $f(\sqrt{2}) = 0$, $f(2) < 0$, and $f(4) > 0$, so there is a second root x_0 between 2 and 4.

Since f has no other sign changes, there are exactly two solutions, and $S = \sqrt{2} + x_0 \approx 1.41 + 3.1 \approx 4.5$, which lies in $[2, 6)$.

Thus, the correct answer is **D**.

22. Arjun and Beth play a game in which they take turns removing one brick or two adjacent bricks from one "wall" among a set of several walls of bricks, with gaps possibly creating new walls. The walls are one brick tall. For example, a set of walls of sizes 4 and 2 can be changed into any of the following by one move:

$$(3,2),\ (2,1,2),\ (4),\ (4,1),\ (2,2),\ \text{or}\ (1,1,2).$$



Arjun plays first, and the player who removes the last brick wins. For which starting configuration is there a strategy that guarantees a win for Beth?

A $(6,1,1)$

B $(6,2,1)$

C $(6,2,2)$

D $(6,3,1)$

E $(6,3,2)$

Solution:

Treat each wall as a Nim-like heap with a Grundy value. A move removes 1 or 2 adjacent bricks, possibly splitting a wall into two, so $g(n) = \text{mex}$ over all resulting XOR values.

Computing, $g(1) = 1, g(2) = 2, g(3) = 3, g(4) = 1, g(5) = 4, g(6) = 3.$

The second player Beth wins exactly when the XOR of the walls' Grundy values is 0. Checking each option, only $(6,2,1)$ gives $g(6) \oplus g(2) \oplus g(1) = 3 \oplus 2 \oplus 1 = 0.$

Thus, the correct answer is **B**.

23. Three balls are randomly and independently tossed into bins numbered with the positive integers so that for each ball, the probability that it is tossed into bin i is 2^{-i} for $i = 1, 2, 3, \ldots$. More than one ball is allowed in each bin. The probability that the balls end up evenly spaced in distinct bins is $\frac{p}{q}$, where p and q are relatively prime positive integers. (For example, the balls are evenly spaced if they are tossed into bins $3, 17,$ and 10.) What is $p + q$?

A 55

B 56

C 57

D 58

E 59

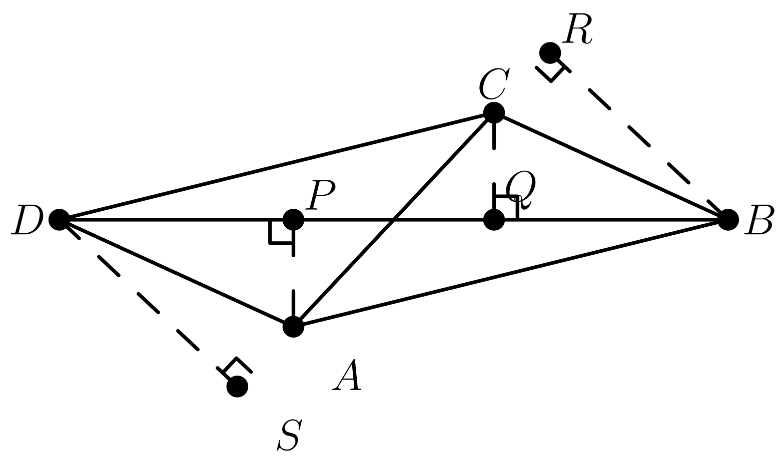
Solution:

Evenly spaced distinct bins form an arithmetic progression $n, n+d, n+2d$ with $n, d \geq 1$. The three labels sum to $3(n+d)$, so a fixed assignment of balls to these bins has probability $2^{-3(n+d)}$.

The three balls can be ordered in $3! = 6$ ways, so the total probability is
$6 \sum_{n\geq 1} \sum_{d \geq 1} 2^{-3(n+d)} = 6 \left(\sum_{n\geq 1} \frac{1}{8^n}\right)^2 = 6 \cdot \frac{1}{7} \cdot \frac{1}{7} = \frac{6}{49}$.
Since $\gcd(6, 49) = 1$, we get $p + q = 6 + 49 = 55.$

Thus, the correct answer is **A**.

24. Let $ABCD$ be a parallelogram with area 15. Points P and Q are the projections of A and C, respectively, onto the line BD; and points R and S are the projections of B and D, respectively, onto the line AC. See the figure, which also shows the relative locations of these points.



Suppose $PQ = 6$ and $RS = 8$, and let d denote the length of $\overline{BD}$, the longer diagonal of $ABCD$. Then d^2 can be written in the form $m + n\sqrt{p}$, where $m, n,$ and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$?

- A 81
- B 89
- C 97
- D 105
- E 113

Solution:

Let the diagonals meet at O at angle θ. The feet of the perpendiculars from A and C to BD are symmetric about O, so $PQ = AC\cos\theta = 6$; likewise $RS = BD\cos\theta = 8$.

The parallelogram's area is $\frac{1}{2} \cdot AC \cdot BD\sin\theta = 15$, so $AC \cdot BD\sin\theta = 30$. Then $\dfrac{48\sin\theta}{\cos^2\theta} = 30$, giving $\dfrac{\sin\theta}{\cos^2\theta} = \dfrac{5}{8}$.

Writing $s = \sin\theta, 8s = 5(1 - s^2)$ gives $s = \dfrac{-4 + \sqrt{41}}{5}$, so $\cos^2\theta = 1 - s^2 = \dfrac{8(\sqrt{41} - 4)}{25}$.

Then $d^2 = BD^2 = \dfrac{64}{\cos^2\theta} = 8(\sqrt{41} + 4) = 32 + 8\sqrt{41}$, so $m + n + p = 32 + 8 + 41 = 81.$

Thus, the correct answer is **A**.

25. Let S be the set of lattice points in the coordinate plane, both of whose coordinates are integers between 1 and 30, inclusive. Exactly 300 points in S lie on or below a line with equation $y = mx$. The possible values of m lie in an interval of length $\dfrac{a}{b}$, where a and b are relatively prime positive integers. What is $a + b$?

A 31

B 47

C 62

D 72

E 85

Solution:

For slope m, column x (with $1 \le x \le 30$) contributes $\min(30, \lfloor mx \rfloor)$ points on or below $y = mx$, and we need the total to equal 300.

The count is a step function of m that jumps at fractions $\frac{y}{x}$. Sweeping through these breakpoints, the count equals 300 for m in a single interval whose endpoints are consecutive such slopes.

That interval runs from $m = \dfrac{2}{3}$ up to $m = \dfrac{19}{28}$, of length $\dfrac{19}{28} - \dfrac{2}{3} = \dfrac{57 - 56}{84} = \dfrac{1}{84}$.

Since $\gcd(1, 84) = 1, a + b = 1 + 84 = 85.$

Thus, the correct answer is **E**.

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