# 2021 AMC 12A Spring Solutions

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1. What is the value of

$$2^{1+2+3} - \left(2^1 + 2^2 + 2^3\right)?$$

- A $0$
- B $50$
- C $52$
- D $54$
- E $57$

### Solution:

The exponent in the first term is $1+2+3=6$, so the first term is $2^6 = 64$. The parenthesized sum is $2+4+8=14$. Therefore the value is $64-14=50$.

Thus, the correct answer is **B**.

**2.** Under what conditions is $\sqrt{a^2+b^2} = a+b$ true, where $a$ and $b$ are real numbers?

- A It is never true.
- B It is true if and only if $ab = 0.$
- C It is true if and only if $a + b \geq 0.$
- D It is true if and only if $ab = 0$ and $a + b \geq 0.$
- E It is always true.

## Solution:

Because $\sqrt{a^2+b^2}$ is never negative, equality requires $a+b \geq 0.$ Squaring both sides gives $a^2+b^2 = (a+b)^2 = a^2+2ab+b^2,$ which simplifies to $2ab = 0,$ i.e. $ab = 0.$

Conversely, if $ab = 0$ then $a^2+b^2 = (a+b)^2,$ and if additionally $a+b \geq 0$ then $\sqrt{a^2+b^2} = |a+b| = a+b.$ So both conditions together are exactly what is needed.

Thus, the correct answer is **D**.

3. The sum of two natural numbers is $17{,}402$. One of the two numbers is divisible by $10$. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

- A $10{,}272$
- B $11{,}700$
- C $13{,}362$
- D $14{,}238$
- E $15{,}426$

## Solution:

The larger number ends in $0$, and erasing that digit divides it by $10$ to give the smaller number. So the larger number is $10$ times the smaller. Writing the smaller number as $x$, the sum is $x + 10x = 11x = 17{,}402$, giving $x = 1{,}582$.

The two numbers are $1{,}582$ and $15{,}820$, whose difference is $15{,}820 - 1{,}582 = 14{,}238$.

Thus, the correct answer is **D**.

**4.** Tom has a collection of $13$ snakes, $4$ of which are purple and $5$ of which are happy. He observes that

- all of his happy snakes can add,
- none of his purple snakes can subtract, and
- all of his snakes that can't subtract also can't add.

Which of these conclusions can be drawn about Tom's snakes?

A Purple snakes can add.

B Purple snakes are happy.

C Snakes that can add are purple.

D Happy snakes are not purple.

E Happy snakes can't subtract.

## Solution:

A purple snake cannot subtract, and any snake that cannot subtract also cannot add. So every purple snake cannot add.

Every happy snake can add. Since purple snakes cannot add, no happy snake can be purple; that is, happy snakes are not purple.

Thus, the correct answer is **D**.

**5.** When a student multiplied the number $66$ by the repeating decimal

$$1.\overline{ab} = 1.ababab\ldots,$$

where $a$ and $b$ are digits, he did not notice the notation and just multiplied $66$ by the terminating decimal $1.ab$. Later he found that his answer was $0.5$ less than the correct answer.

What is the two-digit integer $\overline{ab}$?

A $15$

B $30$

C $45$

D $60$

E $75$

## Solution:

Let $n = \overline{ab}$ be the two-digit integer. Then $1.\overline{ab} = 1 + \frac{n}{99}$ while the terminating value is $1.ab = 1 + \frac{n}{100}$. The correct product minus the student's product is

$$66\left(\frac{n}{99} - \frac{n}{100}\right) = 66 \cdot \frac{n}{9900} = \frac{n}{150}.$$

Setting $\frac{n}{150} = 0.5$ gives $n = 75$.

Thus, the correct answer is **E**.

**6.** A deck of cards has only red cards and black cards. The probability of a randomly chosen card being red is $\frac{1}{3}$. When $4$ black cards are added to the deck, the probability of choosing red becomes $\frac{1}{4}$. How many cards were in the deck originally?

A $6$

B $9$

C $12$

D $15$

E $18$

## Solution:

Let $r$ be the number of red cards and $t$ the total. From $\frac{r}{t} = \frac{1}{3}$ we get $t = 3r$. After adding $4$ black cards, $\frac{r}{t+4} = \frac{1}{4}$, so $t + 4 = 4r$.

Substituting $t = 3r$ gives $3r + 4 = 4r$, so $r = 4$ and $t = 12$.

Thus, the correct answer is **C**.

**7.** What is the least possible value of $(xy-1)^2+(x+y)^2$ for real numbers $x$ and $y$?

A $0$

B $\frac{1}{4}$

C $\frac{1}{2}$

D $1$

E $2$

## Solution:

Expanding,

$$(xy-1)^2+(x+y)^2 = x^2y^2-2xy+1+x^2+2xy+y^2 = x^2y^2+x^2+y^2+1.$$

This factors as $(x^2+1)(y^2+1)$.

Each factor is at least $1$, so the product is at least $1$, with equality when $x=y=0$.

Thus, the correct answer is **D**.

**8.** A sequence of numbers is defined by $D_0 = 0, D_1 = 0, D_2 = 1$, and $D_n = D_{n-1} + D_{n-3}$ for $n \geq 3$. What are the parities (evenness or oddness) of the triple of numbers $(D_{2021}, D_{2022}, D_{2023})$, where $E$ denotes even and $O$ denotes odd?

A $(O, E, O)$

B $(E, E, O)$

C $(E, O, E)$

D $(O, O, E)$

E $(O, O, O)$

## Solution:

Working modulo $2$, the terms $D_0, D_1, D_2, \dots$ have parities

$$E, E, O, O, O, E, O, E, E, O, O, O, E, O, \dots$$

which repeat with period $7$ starting from $D_0$ (indeed $D_7, D_8, D_9$ have the same parities $E, E, O$ as $D_0, D_1, D_2$).

Since $2021 \equiv 5, 2022 \equiv 6$, and $2023 \equiv 0 \pmod 7$, the parities match those of $D_5, D_6, D_0$, namely $E, O, E$.

Thus, the correct answer is **C**.

**9.** Which of the following is equivalent to

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})?$$

A $3^{127}+2^{127}$

B $3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63}$

C $3^{128}-2^{128}$

D $3^{128}+2^{128}$

E $5^{127}$

## Solution:

Since $3-2=1$, multiplying the product by $3-2$ does not change it. Then

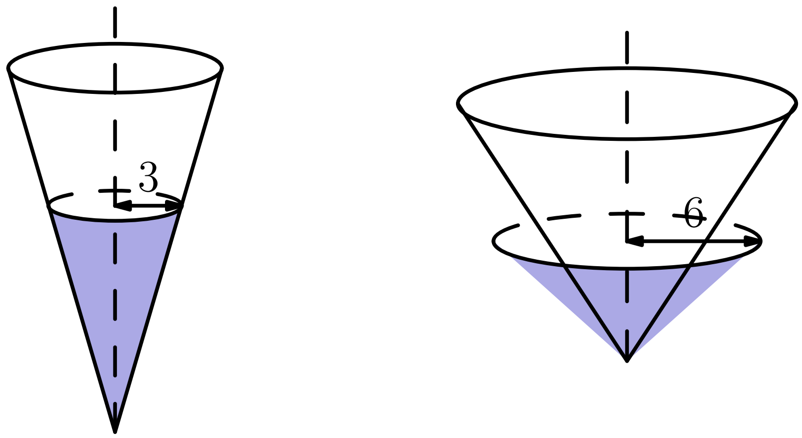
$$(3-2)(3+2)=3^2-2^2,$$

and multiplying by the next factor $(3^2+2^2)$ gives $3^4-2^4$, and so on. Each step doubles the exponent.

After using all seven factors, the product telescopes to $3^{128}-2^{128}$.

Thus, the correct answer is **C**.

**10.** Two right circular cones with vertices facing down as shown in the figure below contain the same amount of liquid. The radii of the tops of the liquid surfaces are $3$ cm and $6$ cm. Into each cone is dropped a spherical marble of radius $1$ cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?




A $1:1$

B $47:43$

C $2:1$

D $40:13$

E $4:1$

## Solution:

The liquid in each cone forms a smaller cone similar to the container. Let the narrow liquid cone have radius $3$ and height $h_1$, and the wide one radius $6$ and height $h_2$. Equal volumes give $\frac{1}{3}\pi \cdot 9 \cdot h_1 = \frac{1}{3}\pi \cdot 36 \cdot h_2$, so $h_1 = 4h_2$.

Dropping the marble raises the volume by the same amount $\Delta V = \frac{4}{3}\pi$ in each cone, and both start with the same volume $V$. Because a cone's volume scales as the cube of its height, the new height is $h\sqrt[3]{1+\Delta V/V}$, so each rise equals $h\left(\sqrt[3]{1+\Delta V/V}-1\right)$. This factor is identical for the two cones, so the rises are in the ratio $h_1 : h_2 = 4:1$.

Thus, the correct answer is **E**.

**11.** A laser is placed at the point $(3, 5)$. The laser beam travels in a straight line. Larry wants the beam to hit and bounce off the $y$-axis, then hit and bounce off the $x$-axis, then hit the point $(7, 5)$. What is the total distance the beam will travel along this path?

A $2\sqrt{10}$

B $5\sqrt{2}$

C $10\sqrt{2}$

D $15\sqrt{2}$

E $10\sqrt{5}$

## Solution:

Reflecting the path at each bounce turns it into a single straight segment. Reflect the start $(3, 5)$ across the $y$-axis to $(-3, 5)$, and reflect the target $(7, 5)$ across the $x$-axis to $(7, -5)$. The total travel distance equals the straight-line distance between these two images:

$$\sqrt{(-3-7)^2 + (5-(-5))^2} = \sqrt{100+100} = 10\sqrt{2}.$$

Thus, the correct answer is **C**.

**12.** All the roots of the polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of $B$?

A $-88$

B $-80$

C $-64$

D $-41$

E $-40$

## Solution:

By Vieta's formulas the six roots sum to $10$ (the negative of the $z^5$ coefficient) and multiply to $16.$ Six positive integers with sum $10$ and product $16$ must be $2, 2, 2, 2, 1, 1.$

So the polynomial is $(z-1)^2(z-2)^4$. Expanding,

$$(z^2 - 2z + 1)(z^4 - 8z^3 + 24z^2 - 32z + 16) = z^6 - 10z^5 + 41z^4 - 88z^3 + 104z^2 - 64z + 16.$$

The coefficient of $z^3$ is $B = -88.$

Thus, the correct answer is **A**.

**13.** Of the following complex numbers $z$, which one has the property that $z^5$ has the greatest real part?

A $-2$

B $-\sqrt{3}+i$

C $-\sqrt{2}+\sqrt{2}\,i$

D $-1+\sqrt{3}\,i$

E $2i$

## Solution:

Each listed number has modulus $2$, so $z^5$ has modulus $32$, and its real part is $32\cos(5\theta)$, where $\theta$ is the argument of $z$. The arguments are $180°, 150°, 135°, 120°$, and $90°$.

Multiplying by $5$ gives $900° \equiv 180°, 750° \equiv 30°, 675° \equiv 315°, 600° \equiv 240°$, and $450° \equiv 90°$. The largest cosine is $\cos 30°$, from $z = -\sqrt{3}+i$, giving real part $16\sqrt{3}$.

Thus, the correct answer is **B**.

**14.** What is the value of

$$\left(\sum_{k=1}^{20} \log_{5^k} 3^{k^2}\right) \cdot \left(\sum_{k=1}^{100} \log_{9^k} 25^k\right)?$$

- A $21$
- B $100 \log_5 3$
- C $200 \log_3 5$
- D $2{,}200$
- E $21{,}000$

## Solution:

For the first sum, $\log_{5^k} 3^{k^2} = \dfrac{k^2}{k} \log_5 3 = k \log_5 3$, so

$$\sum_{k=1}^{20} k \log_5 3 = \frac{20 \cdot 21}{2} \log_5 3 = 210 \log_5 3.$$

For the second sum, $\log_{9^k} 25^k = \log_9 25 = \log_3 5$, independent of $k$, so the sum is $100 \log_3 5$.

Since $\log_5 3 \cdot \log_3 5 = 1$, the product is $210 \cdot 100 = 21{,}000$.

Thus, the correct answer is **E**.

**15.** A choir director must select a group of singers from among his $6$ tenors and $8$ basses. The only requirements are that the difference between the number of tenors and basses must be a multiple of $4,$ and the group must have at least one singer. Let $N$ be the number of groups that can be selected. What is the remainder when $N$ is divided by $100?$

A $47$

B $48$

C $83$

D $95$

E $96$

## Solution:

Choosing $t$ tenors and $b$ basses is weighted by $\binom{6}{t}\binom{8}{b}$. To keep only $t-b\equiv 0 \pmod 4$, apply a roots of unity filter with $\omega = i$ :

$$N+1=\frac{1}{4}\sum_{j=0}^{3}(1+i^{j})^{6}\,(1+i^{-j})^{8}.$$

The $j=0$ term is $2^6\cdot 2^8=16384$. The $j=2$ term has factor $(1+i^2)^6=0$. The $j=1$ and $j=3$ terms are $-128i$ and $128i$, which cancel. So the sum is $16384$, and $\dfrac{16384}{4}=4096.$

This count includes the empty group, so $N=4096-1=4095$, and $N\equiv 95 \pmod{100}$.

Thus, the correct answer is **D**.

**16.** In the following list of numbers, the integer $n$ appears $n$ times in the list for $1 \leq n \leq 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, 200, \ldots, 200$$

What is the median of the numbers in this list?

A $100.5$

B $134$

C $142$

D $150.5$

E $167$

## Solution:

The list has $1 + 2 + \cdots + 200 = \dfrac{200 \cdot 201}{2} = 20100$ terms, so the median is the average of the $10050$th and $10051$st terms.

The value $n$ occupies positions up to $\dfrac{n(n+1)}{2}$. Since $\dfrac{141 \cdot 142}{2} = 10011$ and $\dfrac{142 \cdot 143}{2} = 10153$, positions $10012$ through $10153$ all equal $142$. Both middle positions fall in this block, so the median is $142$.

Thus, the correct answer is **C**.

**17.** Trapezoid $ABCD$ has $AB \parallel CD, BC = CD = 43,$ and $AD \perp BD.$ Let $O$ be the intersection of the diagonals $AC$ and $BD,$ and let $P$ be the midpoint of $BD.$ Given that $OP = 11,$ the length $AD$ can be written in the form $m\sqrt{n},$ where $m$ and $n$ are positive integers and $n$ is not divisible by the square of any prime. What is $m + n$?

A $65$

B $132$

C $157$

D $194$

E $215$

## Solution:

Place $D = (0, 0)$ with $B = (b, 0)$ on one axis and $A = (0, a)$ on the other, so that $AD \perp BD.$ Since $CD \parallel AB,$ write $C = t(b, -a)$ for some $t.$ Then $CD = t\sqrt{a^2 + b^2}$ and $BC^2 = b^2(1 - t)^2 + t^2a^2.$ Setting $BC = CD$ gives $t^2 = (1 - t)^2,$ so $t = \frac{1}{2}.$

Thus $C = \left(\frac{b}{2}, -\frac{a}{2}\right)$, and $CD = 43$ gives $a^2 + b^2 = 4 \cdot 43^2 = 7396.$ The diagonal $AC$ meets $BD$ (the $x$-axis) at $O = \left(\frac{b}{3}, 0\right)$, while $P = \left(\frac{b}{2}, 0\right)$. Hence $OP = \frac{b}{6} = 11,$ so $b = 66.$

Then $a^2 = 7396 - 66^2 = 3040,$ so $AD = a = \sqrt{3040} = 4\sqrt{190}.$ With $m = 4$ and $n = 190,$ we get $m + n = 194.$

Thus, the correct answer is **D**.

**18.** Let $f$ be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers $a$ and $b$. Suppose that $f$ also has the property that $f(p) = p$ for every prime number $p$. For which of the following numbers $x$ is $f(x) < 0$?

A $\frac{17}{32}$

B $\frac{11}{16}$

C $\frac{7}{9}$

D $\frac{7}{6}$

E $\frac{25}{11}$

## Solution:

The functional equation makes $f$ completely additive: for $x = \prod p^{e_p}$, we have $f(x) = \sum e_p\, f(p) = \sum e_p\, p$, where a prime in the denominator contributes a negative exponent (since $f(1/p) = -p$).

Evaluating: $f\left(\frac{17}{32}\right) = 17 - 5 \cdot 2 = 7$, $f\left(\frac{11}{16}\right) = 11 - 4 \cdot 2 = 3$, $f\left(\frac{7}{9}\right) = 7 - 2 \cdot 3 = 1$, $f\left(\frac{7}{6}\right) = 7 - 2 - 3 = 2$, and $f\left(\frac{25}{11}\right) = 2 \cdot 5 - 11 = -1$. Only the last is negative.

Thus, the correct answer is **E**.

**19.** How many solutions does the equation

$$\sin\left(\frac{\pi}{2}\cos x\right) = \cos\left(\frac{\pi}{2}\sin x\right)$$

have in the closed interval $[0, \pi]$?

A $0$

B $1$

C $2$

D $3$

E $4$

## Solution:

Write the right side as $\cos\left(\frac{\pi}{2}\sin x\right) = \sin\left(\frac{\pi}{2} - \frac{\pi}{2}\sin x\right)$. Equal sines require either

$$\frac{\pi}{2}\cos x = \frac{\pi}{2}(1 - \sin x) + 2\pi k \quad \text{or} \quad \frac{\pi}{2}\cos x = \pi - \frac{\pi}{2}(1 - \sin x) + 2\pi k.$$

The first reduces to $\cos x + \sin x = 1 + 4k$; since $\cos x + \sin x \in [-\sqrt{2}, \sqrt{2}]$, only $k = 0$ works, giving $\cos x + \sin x = 1$, with solutions $x = 0$ and $x = \frac{\pi}{2}$ in $[0, \pi]$. The second reduces to $\cos x - \sin x = 1$, whose only solution in $[0, \pi]$ is $x = 0$.

The distinct solutions are $x = 0$ and $x = \frac{\pi}{2}$, for a total of $2$.

Thus, the correct answer is **C**.

**20.** Suppose that on a parabola with vertex $V$ and a focus $F$ there exists a point $A$ such that $AF = 20$ and $AV = 21$. What is the sum of all possible values of the length $FV$?

A $13$

B $\frac{40}{3}$

C $\frac{41}{3}$

D $14$

E $\frac{43}{3}$

## Solution:

Let $V = (0, 0)$, focus $F = (0, f)$, and directrix $y = -f$, where $f = FV$. A point $A = (x, y)$ on the parabola satisfies $x^2 = 4fy$ and $AF = y + f = 20$, so $y = 20 - f$. Also $AV^2 = x^2 + y^2 = 4fy + y^2 = 441$.

Substituting $y = 20 - f$ :

$$4f(20 - f) + (20 - f)^2 = 441 \implies 3f^2 - 40f + 41 = 0.$$

By Vieta's formulas, the sum of the two possible values of $f$ is $\frac{40}{3}$.

Thus, the correct answer is **B**.

**21.** The five solutions to the equation

$$(z-1)(z^2+2z+4)(z^2+4z+6)=0$$

may be written in the form $x_k + y_k i$ for $1 \le k \le 5$, where $x_k$ and $y_k$ are real. Let $E$ be the unique ellipse that passes through the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$, and $(x_5, y_5)$. The eccentricity of $E$ can be written in the form $\sqrt{\frac{m}{n}}$, where $m$ and $n$ are relatively prime positive integers. What is $m+n$?

(Recall that the eccentricity of an ellipse $E$ is the ratio $\frac{c}{a}$, where $2a$ is the length of the major axis of $E$ and $2c$ is the distance between its two foci.)

A $7$

B $9$

C $11$

D $13$

E $15$

## Solution:

The roots are $z=1, z=-1\pm i\sqrt{3}$, and $z=-2\pm i\sqrt{2}$, giving the points $(1,0), (-1,\pm\sqrt{3})$, and $(-2,\pm\sqrt{2})$. By symmetry about the $x$-axis, the ellipse has the form $Ax^2+Cy^2+Dx+F=0$.

Substituting the points yields $5x^2+6y^2+9x-14=0$. Completing the square gives

$$5\left(x+\tfrac{9}{10}\right)^2+6y^2=\tfrac{361}{20},$$

so $a^2=\frac{361}{100}$ (along $x$) and $b^2=\frac{361}{120}$. Then $e^2=1-\dfrac{b^2}{a^2}=1-\dfrac{100}{120}=\dfrac{1}{6}$, so $e=\sqrt{\frac{1}{6}}$.

With $m=1$ and $n=6$, we get $m+n=7$.

Thus, the correct answer is **A**.

**22.** Suppose that the roots of the polynomial $P(x) = x^3 + ax^2 + bx + c$ are $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}$, and $\cos\frac{6\pi}{7}$, where angles are in radians. What is $abc$?

A $-\dfrac{3}{49}$

B $-\dfrac{1}{28}$

C $\dfrac{\sqrt[3]{7}}{64}$

D $\dfrac{1}{32}$

E $\dfrac{1}{28}$

## Solution:

The numbers $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{6\pi}{7}$ are the three roots of $8x^3 + 4x^2 - 4x - 1 = 0$. Dividing by $8$ puts it in monic form:

$$x^3 + \tfrac{1}{2}x^2 - \tfrac{1}{2}x - \tfrac{1}{8} = 0.$$

Matching coefficients, $a = \frac{1}{2}, b = -\frac{1}{2}, c = -\frac{1}{8}$. Therefore $abc = \frac{1}{2} \cdot \left(-\frac{1}{2}\right) \cdot \left(-\frac{1}{8}\right) = \frac{1}{32}$.

Thus, the correct answer is **D**.

**23.** Frieda the frog begins a sequence of hops on a $3 \times 3$ grid of squares, moving one square on each hop and choosing at random the direction of each hop: up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example, if Frieda begins in the center square and makes two hops "up," the first hop places her in the top row middle square, and the second hop causes her to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

A $\frac{9}{16}$

B $\frac{5}{8}$

C $\frac{3}{4}$

D $\frac{25}{32}$

E $\frac{13}{16}$

## Solution:

Classify squares as center $C$, edge-middle $E$, or corner (absorbing). From $C$, every hop lands on an $E$ square. From an $E$ square, two of the four neighbors are corners, one is the center, and one is another $E$ square, so $P(\text{corner}) = \frac{1}{2}, P(\text{center}) = \frac{1}{4}, P(E) = \frac{1}{4}$.

Let $a_n$ be the probability of reaching a corner within $n$ hops starting from an edge square, and $c_n = a_{n-1}$ the probability starting from the center (the first hop always goes to an edge). Then $a_1 = \frac{1}{2}$ and $a_n = \frac{1}{2} + \frac{1}{4}c_{n-1} + \frac{1}{4}a_{n-1}$. Computing: $a_2 = \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{5}{8}$, and

$$a_3 = \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{5}{8} = \frac{25}{32}.$$

Starting from the center with four hops available, the probability equals $a_3 = \frac{25}{32}$ (the first hop reaches an edge, leaving three hops).

Thus, the correct answer is **D**.

**24.** Semicircle $\Gamma$ has diameter $AB$ of length $14$. Circle $\Omega$ lies tangent to $AB$ at a point $P$ and intersects $\Gamma$ at points $Q$ and $R$. If $QR = 3\sqrt{3}$ and $\angle QPR = 60^\circ$, then the area of $\triangle PQR$ is $\dfrac{a\sqrt{b}}{c}$, where $a$ and $c$ are relatively prime positive integers and $b$ is a positive integer not divisible by the square of any prime. What is $a + b + c$?

A $110$

B $114$

C $118$

D $122$

E $126$

## Solution:

In circle $\Omega$, the chord $QR$ subtends the inscribed angle $\angle QPR = 60^\circ$, so $QR = 2r\sin 60^\circ$, giving $3\sqrt{3} = r\sqrt{3}$, hence $r = 3$.

Place $A = (-7, 0), B = (7, 0)$, with $\Gamma : x^2 + y^2 = 49$ (upper half). Since $\Omega$ is tangent to $AB$ at $P = (p, 0)$, its center is $(p, 3)$. Subtracting the two circle equations gives the line $QR$, and the distance from the center $(p, 3)$ to $QR$ must equal $r\cos 60^\circ = \frac{3}{2}$. This yields $(p^2 - 31)^2 = 9p^2 + 81$, so $p^2 = 16$ (the root $p^2 = 55$ places $P$ outside $AB$).

With $p^2 = 16$, the distance from $P$ to line $QR$ is $\dfrac{49 - p^2}{\sqrt{4p^2 + 36}} = \dfrac{33}{10}$. Thus

$$[\triangle PQR] = \tfrac{1}{2} \cdot QR \cdot d = \tfrac{1}{2} \cdot 3\sqrt{3} \cdot \tfrac{33}{10} = \frac{99\sqrt{3}}{20}.$$

So $a = 99, b = 3, c = 20$, and $a + b + c = 122$.

Thus, the correct answer is **D**.

**25.** Let $d(n)$ denote the number of positive integers that divide $n$, including $1$ and $n$. For example, $d(1) = 1, d(2) = 2$, and $d(12) = 6$. (This function is known as the divisor function.) Let

$$f(n) = \frac{d(n)}{\sqrt[3]{n}}.$$

There is a unique positive integer $N$ such that $f(N) > f(n)$ for all positive integers $n \neq N$. What is the sum of the digits of $N$?

A $5$

B $6$

C $7$

D $8$

E $9$

## Solution:

Since $f(n) = \dfrac{d(n)}{n^{1/3}}$ is multiplicative, its value factors over prime powers as a product of terms $\dfrac{e+1}{p^{e/3}}$ for each prime power $p^e \parallel n$. We maximize each term separately.

For $p = 2$, the ratio $\dfrac{e+1}{2^{e/3}}$ is largest at $e = 3$ (value $2$). For $p = 3$, it peaks at $e = 2$; for $p = 5$ and $p = 7$, at $e = 1$; and for every prime $p \geq 11$, the best choice is $e = 0$ (the ratio already drops below $1$ at $e = 1$).

Hence $N = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$, whose digit sum is $2 + 5 + 2 + 0 = 9$.

Thus, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/2021A