2020 AMC 12B Solutions

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1. What is the value in simplest form of the following expression?

$$\sqrt{1}+\sqrt{1+3}+\sqrt{1+3+5}+\sqrt{1+3+5+7}$$

A 5

B $4+\sqrt{7}+\sqrt{10}$

C 10

D 15

E $4+3\sqrt{3}+2\sqrt{5}+\sqrt{7}$

Solution:

The sum of the first k odd numbers equals k^2, so each radicand is a perfect square:

$$\sqrt{1}+\sqrt{4}+\sqrt{9}+\sqrt{16}=1+2+3+4=10.$$

Thus, the correct answer is **C**.

2. What is the value of the following expression?

$$\frac{100^2 - 7^2}{70^2 - 11^2} \cdot \frac{(70-11)(70+11)}{(100-7)(100+7)}$$

- A 1
- B $\frac{9951}{9950}$
- C $\frac{4780}{4779}$
- D $\frac{108}{107}$
- E $\frac{81}{80}$

Solution:

Using the difference of squares, $100^2 - 7^2 = (100-7)(100+7)$ and $70^2 - 11^2 = (70-11)(70+11)$. The expression becomes

$$\frac{(100-7)(100+7)}{(70-11)(70+11)} \cdot \frac{(70-11)(70+11)}{(100-7)(100+7)} = 1.$$

Thus, the correct answer is **A**.

3. The ratio of w to x is $4:3$, the ratio of y to z is $3:2$, and the ratio of z to x is $1:6$. What is the ratio of w to y?

A $4:3$

B $3:2$

C $8:3$

D $4:1$

E $16:3$

Solution:

Let $x=6$. From $z:x=1:6$, we get $z=1$. From $w:x=4:3$, we get $w=\frac{4}{3}\cdot 6=8$. From $y:z=3:2$, we get $y=\frac{3}{2}\cdot 1=\frac{3}{2}$.

Therefore $w:y=8:\frac{3}{2}=16:3$.

Thus, the correct answer is **E**.

4. The acute angles of a right triangle are a° and b°, where $a > b$ and both a and b are prime numbers. What is the least possible value of b?

- A 2
- B 3
- C 5
- D 7
- E 11

Solution:

Since the angles are complementary, $a + b = 90.$ To minimize $b,$ try small primes and require $90 - b$ to be prime as well.

For $b = 2, 3, 5,$ the value $90 - b = 88, 87, 85$ is not prime. For $b = 7,$ we get $90 - 7 = 83,$ which is prime. So the least possible value is $b = 7.$

Thus, the correct answer is **D**.

5. Teams A and B are playing in a basketball league where each game results in a win for one team and a loss for the other team. Team A has won $\frac{2}{3}$ of its games and team B has won $\frac{5}{8}$ of its games. Also, team B has won 7 more games and lost 7 more games than team A. How many games has team A played?

- A 21
- B 27
- C 42
- D 48
- E 63

Solution:

Let a be the number of games team A played and b the number team B played. Team A wins $\frac{2}{3}a$ and loses $\frac{1}{3}a$; team B wins $\frac{5}{8}b$ and loses $\frac{3}{8}b$. The conditions give

$$\frac{5}{8}b = \frac{2}{3}a + 7 \quad \text{and} \quad \frac{3}{8}b = \frac{1}{3}a + 7.$$

Subtracting the equations gives $\frac{1}{4}b = \frac{1}{3}a$, so $b = \frac{4}{3}a$. Substituting into the loss equation: $\frac{3}{8} \cdot \frac{4}{3}a = \frac{1}{3}a + 7$, i.e. $\frac{1}{2}a = \frac{1}{3}a + 7$, so $\frac{1}{6}a = 7$ and $a = 42$.

Thus, the correct answer is **C**.

6. For all integers $n \geq 9$, the value of

$$\frac{(n+2)! - (n+1)!}{n!}$$

is always which of the following?

A a multiple of 4

B a multiple of 10

C a prime number

D a perfect square

E a perfect cube

Solution:

Factor $(n+1)!$ from the numerator:

$$(n+2)! - (n+1)! = (n+1)!\left[(n+2) - 1\right] = (n+1)!\,(n+1).$$

Dividing by $n!$ leaves $\dfrac{(n+1)!\,(n+1)}{n!} = (n+1)(n+1) = (n+1)^2$, which is always a perfect square.

Thus, the correct answer is **D**.

7. Two nonhorizontal, non-vertical lines in the xy-coordinate plane intersect to form a $45°$ angle. One line has slope equal to 6 times the slope of the other line. What is the greatest possible value of the product of the slopes of the two lines?

A $\frac{1}{6}$

B $\frac{2}{3}$

C $\frac{3}{2}$

D 3

E 6

Solution:

Let the slopes be m and $6m$. The angle between the lines satisfies

$$\left|\frac{6m - m}{1 + 6m^2}\right| = \tan 45° = 1,$$

so $5m = \pm(1 + 6m^2)$, giving $6m^2 - 5m + 1 = 0$ or $6m^2 + 5m + 1 = 0$.

The first yields $m = \frac{1}{2}$ or $m = \frac{1}{3}$; the second yields the negatives of these. The product of the slopes is $6m^2$, which is largest when $m = \frac{1}{2}$, giving $6 \cdot \frac{1}{4} = \frac{3}{2}$.

Thus, the correct answer is **C**.

8. How many ordered pairs of integers (x, y) satisfy the equation

$$x^{2020} + y^2 = 2y?$$

A 1

B 2

C 3

D 4

E infinitely many

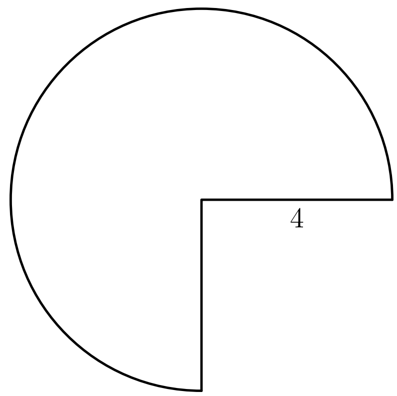
Solution:

Completing the square gives $x^{2020} + (y-1)^2 = 1$. Both terms are nonnegative, so $x^{2020} \leq 1$, forcing $x \in \{-1, 0, 1\}$.

If $x = 0$, then $(y-1)^2 = 1$, giving $y = 0$ or $y = 2$. If $x = \pm 1$, then $x^{2020} = 1$, so $(y-1)^2 = 0$ and $y = 1$. The solutions are $(0,0)$, $(0,2)$, $(1,1)$, and $(-1,1)$ — four in all.

Thus, the correct answer is **D**.

9. A three-quarter sector of a circle of radius 4 inches together with its interior can be rolled up to form the lateral surface of a right circular cone by taping together along the two radii shown. What is the volume of the cone in cubic inches?



A $3\pi\sqrt{5}$

B $4\pi\sqrt{3}$

C $3\pi\sqrt{7}$

D $6\pi\sqrt{3}$

E $6\pi\sqrt{7}$

Solution:

The sector's arc length is $\frac{3}{4} \cdot 2\pi \cdot 4 = 6\pi$, which becomes the base circumference: $2\pi r = 6\pi$, so $r = 3$.

The slant height is the sector radius 4, so the height is $h = \sqrt{4^2 - 3^2} = \sqrt{7}$. The volume is

$$\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 9 \cdot \sqrt{7} = 3\pi\sqrt{7}.$$

Thus, the correct answer is **C**.

10. In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. What is AP?

A $\frac{\sqrt{5}}{12}$

B $\frac{\sqrt{5}}{10}$

C $\frac{\sqrt{5}}{9}$

D $\frac{\sqrt{5}}{8}$

E $\frac{2\sqrt{5}}{15}$

Solution:

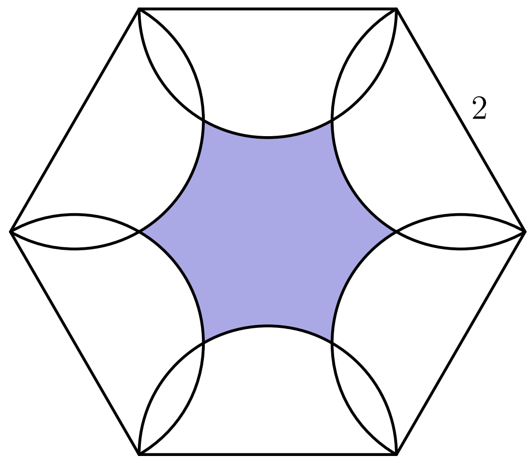
Let $A=(0,0), B=(1,0), C=(1,1), D=(0,1)$. The inscribed circle has center $\left(\frac{1}{2},\frac{1}{2}\right)$ and radius $\frac{1}{2}$, touching $\overline{CD}$ at $M=\left(\frac{1}{2},1\right)$.

Line AM is $y=2x$. Substituting into $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$ gives $20x^2-12x+1=0$, with roots $x=\frac{1}{2}$ (point M) and $x=\frac{1}{10}$ (point P).

So $P=\left(\frac{1}{10},\frac{1}{5}\right)$ and $AP=\sqrt{\left(\frac{1}{10}\right)^2+\left(\frac{1}{5}\right)^2}=\frac{\sqrt{5}}{10}$.

Thus, the correct answer is **B**.

11. As shown in the figure below, six semicircles lie in the interior of a regular hexagon with side length 2 so that the diameters of the semicircles coincide with the sides of the hexagon. What is the area of the shaded region—inside the hexagon but outside all of the semicircles?



A $6\sqrt{3} - 3\pi$

B $\frac{9\sqrt{3}}{2} - 2\pi$

C $\frac{3\sqrt{3}}{2} - \frac{\pi}{3}$

D $3\sqrt{3} - \pi$

E $\frac{9\sqrt{3}}{2} - \pi$

Solution:

The hexagon has area $\frac{3\sqrt{3}}{2} \cdot 2^2 = 6\sqrt{3}$. Each semicircle has radius 1 and area $\frac{\pi}{2}$, totaling 3π.

Adjacent semicircle centers (side midpoints) are a distance $\sqrt{3}$ apart, so each adjacent pair overlaps in a lens of area $2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$. There are six such lenses.

The union of the semicircles is

$$3\pi - 6\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}.$$

Subtracting from the hexagon gives the shaded area $6\sqrt{3} - (\pi + 3\sqrt{3}) = 3\sqrt{3} - \pi$.

Thus, the correct answer is **D**.

12. Let $\overline{AB}$ be a diameter in a circle of radius $5\sqrt{2}$. Let $\overline{CD}$ be a chord in the circle that intersects $\overline{AB}$ at a point E such that $BE = 2\sqrt{5}$ and $\angle AEC = 45^\circ$. What is $CE^2 + DE^2$?

- A 96
- B 98
- C $44\sqrt{5}$
- D $70\sqrt{2}$
- E 100

Solution:

Place the center at the origin with $\overline{AB}$ on the x-axis; the radius is $R = 5\sqrt{2}$, so $R^2 = 50$. Then $E = (x_E, 0)$ with $x_E = R - 2\sqrt{5}$ (its exact value is not needed).

Parametrize the chord as $E + t\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Substituting into $x^2 + y^2 = 50$ gives $t^2 + \sqrt{2}\,x_E\,t + (x_E^2 - 50) = 0$, whose roots are the signed distances t_1, t_2 to C and D. By Vieta, $t_1 + t_2 = -\sqrt{2}\,x_E$ and $t_1 t_2 = x_E^2 - 50$, so

$$CE^2 + DE^2 = t_1^2 + t_2^2 = (t_1 + t_2)^2 - 2t_1t_2 = 2x_E^2 - 2(x_E^2 - 50) = 100.$$

Thus, the correct answer is **E**.

13. Which of the following is the value of $\sqrt{\log_2 6 + \log_3 6}$?

A 1

B $\sqrt{\log_5 6}$

C 2

D $\sqrt{\log_2 3} + \sqrt{\log_3 2}$

E $\sqrt{\log_2 6} + \sqrt{\log_3 6}$

Solution:

Let $a = \log_2 3$, so $\log_3 2 = \frac{1}{a}$. Then

$$\log_2 6 + \log_3 6 = (1 + \log_2 3) + (1 + \log_3 2) = 2 + a + \frac{1}{a}.$$

Meanwhile $\left(\sqrt{\log_2 3} + \sqrt{\log_3 2}\right)^2 = a + \frac{1}{a} + 2\sqrt{a \cdot \frac{1}{a}} = a + \frac{1}{a} + 2$, which equals the expression above.

Taking square roots, $\sqrt{\log_2 6 + \log_3 6} = \sqrt{\log_2 3} + \sqrt{\log_3 2}$.

Thus, the correct answer is **D**.

14. Bela and Jenn play the following game on the closed interval $[0, n]$ of the real number line, where n is a fixed integer greater than 4. They take turns playing, with Bela going first. At his first turn, Bela chooses any real number in the interval $[0, n]$. Thereafter, the player whose turn it is chooses a real number that is more than one unit away from all numbers previously chosen by either player. A player unable to choose such a number loses. Using optimal strategy, which player will win the game?

- A Bela will always win.
- B Jenn will always win.
- C Bela will win if and only if n is odd.
- D Jenn will win if and only if n is odd.
- E Jenn will win if and only if $n > 8$.

Solution:

Bela first plays the midpoint $\frac{n}{2}$. This choice makes the configuration symmetric about the center of the interval.

Thereafter, whenever Jenn picks a number x, Bela responds with its mirror image $n - x$. Since the position was symmetric before Jenn moved and her move is legal, its reflection is also legal and distinct. Thus Bela always has a move whenever Jenn does, so Jenn is the first to be stuck. Bela always wins.

Thus, the correct answer is **A**.

15. There are 10 people standing equally spaced around a circle. Each person knows exactly 3 of the other 9 people: the 2 people standing next to her or him, as well as the person directly across the circle. How many ways are there for the 10 people to split up into 5 pairs so that the members of each pair know each other?

A 11

B 12

C 13

D 14

E 15

Solution:

Label the people 0 through $9.$ Allowed pairings use neighbor edges $(i, i+1)$ or diameter edges $(i, i+5).$ Count perfect matchings by the number of diameter edges used.

Using no diameters, the ten people split into adjacent pairs in 2 ways (all "even" edges or all "odd" edges). Using exactly one diameter, choose it in 5 ways; the remaining two arcs of four people each pair up uniquely, giving $5.$ Using all five diameters gives 1 matching.

Using exactly three diameters accounts for the remaining cases: there are 5 such matchings (two diameters can never be used without forcing an unmatchable odd arc). In total, $2+5+5+1=13.$

Thus, the correct answer is **C**.

16. An urn contains one red ball and one blue ball. A box of extra red and blue balls lies nearby. George performs the following operation four times: he draws a ball from the urn at random and then takes a ball of the same color from the box and returns those two matching balls to the urn. After the four iterations the urn contains six balls. What is the probability that the urn contains three balls of each color?

A $\frac{1}{6}$

B $\frac{1}{5}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

To end with three of each color, exactly two of the four added balls must be red. Consider any sequence of draws. When the urn holds k balls, drawing a particular color with count c has probability $\frac{c}{k}$.

Any ordering with two red and two blue additions gives the same product $\frac{1 \cdot 2 \cdot 1 \cdot 2}{2 \cdot 3 \cdot 4 \cdot 5}$, and there are $\binom{4}{2} = 6$ such orderings, for probability $\frac{6 \cdot 4}{120} = \frac{1}{5}$. (Equivalently, the number of red balls after four steps is uniform on $\{1, 2, 3, 4, 5\}$, so 3 red occurs with probability $\frac{1}{5}$.)

Thus, the correct answer is **B**.

17. How many polynomials of the form $x^5 + ax^4 + bx^3 + cx^2 + dx + 2020$, where $a, b, c,$ and d are real numbers, have the property that whenever r is a root, so is $\dfrac{-1+i\sqrt{3}}{2} \cdot r$? (Note that $i = \sqrt{-1}$.)

- A 0
- B 1
- C 2
- D 3
- E 4

Solution:

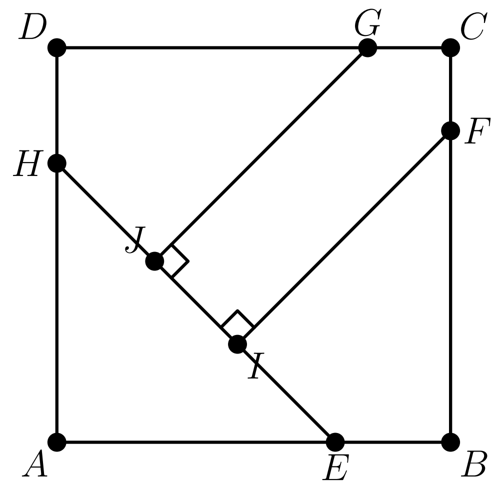
Here $\omega = \frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of unity. Since 0 is not a root, the set of distinct roots is closed under multiplication by ω, so it consists of triples $\{r, \omega r, \omega^2 r\}$ equally spaced in argument. Five roots cannot fill two such triples, so there is exactly one triple, with multiplicities $m_1, m_2, m_3 \geq 1$ summing to 5.

Real coefficients require the root multiset to be closed under conjugation. This is possible only when the triple's arguments are symmetric about the real axis, which happens for the two configurations $\{0°, 120°, 240°\}$ and $\{60°, 180°, 300°\}$.

The product of the roots must equal -2020. In the first configuration the real root is positive, forcing a positive product, which is impossible. In the second, the real root is negative and the product is $-\rho^5$; setting $\rho^5 = 2020$ works, and the two conjugate-symmetric multiplicity patterns $(1, 3, 1)$ and $(2, 1, 2)$ each give a valid polynomial. Hence there are 2.

Thus, the correct answer is **C**.

18. In square $ABCD$, points E and H lie on $\overline{AB}$ and $\overline{DA}$, respectively, so that $AE = AH$. Points F and G lie on $\overline{BC}$ and $\overline{CD}$, respectively, and points I and J lie on $\overline{EH}$ so that $\overline{FI} \perp \overline{EH}$ and $\overline{GJ} \perp \overline{EH}$. See the figure below. Triangle AEH, quadrilateral $BFIE$, quadrilateral $DHJG$, and pentagon $FCGJI$ each has area 1. What is FI^2?



A $\frac{7}{3}$

B $8 - 4\sqrt{2}$

C $1 + \sqrt{2}$

D $\frac{7}{4}\sqrt{2}$

E $2\sqrt{2}$

Solution:

The four regions have total area 4, so the square has side 2. Put $A = (0,0)$, $B = (2,0)$, $C = (2,2)$, $D = (0,2)$. Since $\triangle AEH$ is an isosceles right triangle with area 1, we get $AE = AH = \sqrt{2}$, so $E = (\sqrt{2}, 0)$ and $H = (0, \sqrt{2})$. Line EH is $x + y = \sqrt{2}$.

Let $F = (2, t)$. Its perpendicular distance to line EH is $FI = \frac{2+t-\sqrt{2}}{\sqrt{2}}$. Writing $s = FI/\sqrt{2} = \frac{2+t-\sqrt{2}}{2}$, the quadrilateral $BFIE$ has area 1. Solving this condition gives $s^2 = 4 - 2\sqrt{2}$.

Then $FI^2 = 2s^2 = 8 - 4\sqrt{2}$.

Thus, the correct answer is **B**.

19. Square $ABCD$ in the coordinate plane has vertices at the points $A(1,1), B(-1,1)$, $C(-1,-1)$, and $D(1,-1)$. Consider the following four transformations:

L, a rotation of 90° counterclockwise around the origin;

R, a rotation of 90° clockwise around the origin;

H, a reflection across the x-axis; and

V, a reflection across the y-axis.

Each of these transformations maps the square onto itself, but the positions of the labeled vertices will change. For example, applying R and then V would send the vertex A at $(1,1)$ to $(-1,-1)$ and would send the vertex B at $(-1,1)$ to itself. How many sequences of 20 transformations chosen from $\{L, R, H, V\}$ will send all of the labeled vertices back to their original positions? (For example, R, R, V, H is one sequence of 4 transformations that will send the vertices back to their original positions.)

A 2^{37}

B $3 \cdot 2^{36}$

C 2^{38}

D $3 \cdot 2^{37}$

E 2^{39}

Solution:

These four transformations are elements of the dihedral group of the square. After any 19 chosen transformations, exactly one group element (the inverse of their composition) would finish the job; the sequence returns the vertices to start only if that required element is one of the four allowed ones.

Track a single vertex, say A. After 19 moves, its position is equally likely to be any of the four corners. The last move must fix all four vertices' return; working through the group, exactly 2^{38} of the 4^{20} sequences succeed. (A character computation on the dihedral group gives $\frac{1}{8}\left(4^{20}+4^{20}\right)=4^{19}=2^{38}$.)

Thus, the correct answer is **C**.

20. Two different cubes of the same size are to be painted, with the color of each face being chosen independently and at random to be either black or white. What is the probability that after they are painted, the cubes can be rotated to be identical in appearance?

A $\frac{9}{64}$

B $\frac{289}{2048}$

C $\frac{73}{512}$

D $\frac{147}{1024}$

E $\frac{589}{4096}$

Solution:

For a fixed first cube, the number of second cubes matching it (up to rotation) equals the size of its rotation orbit. So the desired probability is $\frac{1}{64^2}\sum_{\text{orbits}}(\text{orbit size})^2$.

Grouping by black-face count, the orbit sizes are: 0 or 6 black $\rightarrow 1$; 1 or 5 black $\rightarrow 6$; 2 or 4 black $\rightarrow 3$ (opposite) and 12 (adjacent); 3 black $\rightarrow 8$ (corner) and 12 (band). Then

$$\sum(\text{orbit size})^2 = 1 + 36 + (9 + 144) + (64 + 144) + (9 + 144) + 36 + 1 = 588.$$

The probability is $\frac{588}{4096} = \frac{147}{1024}$.

Thus, the correct answer is **D**.

21. How many positive integers n satisfy

$$\frac{n+1000}{70} = \lfloor \sqrt{n} \rfloor?$$

(Recall that $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

- A 2
- B 4
- C 6
- D 30
- E 32

Solution:

The right side is an integer, so let $k = \lfloor \sqrt{n} \rfloor$. Then $n = 70k - 1000$, and $k = \lfloor \sqrt{n} \rfloor$ requires $k^2 \le n < (k+1)^2$.

The lower bound $k^2 \le 70k - 1000$ gives $k^2 - 70k + 1000 \le 0$, i.e. $20 \le k \le 50$. The upper bound $70k - 1000 < (k+1)^2$ gives $k^2 - 68k + 1001 > 0$, i.e. $k \le 21$ or $k \ge 47$.

Intersecting, $k \in \{20, 21, 47, 48, 49, 50\}$, giving 6 values of n.

Thus, the correct answer is **C**.

22. What is the maximum value of

$$\frac{\left(2^t - 3t\right) t}{4^t}$$

for real values of t?

A $\frac{1}{16}$

B $\frac{1}{15}$

C $\frac{1}{12}$

D $\frac{1}{10}$

E $\frac{1}{9}$

Solution:

Split the fraction: $\frac{(2^t - 3t)t}{4^t} = \frac{t}{2^t} - \frac{3t^2}{4^t}$. Let $u = \frac{t}{2^t}$, so $\frac{t^2}{4^t} = u^2$ and the expression is $u - 3u^2$.

This parabola has maximum at $u = \frac{1}{6}$, with value $\frac{1}{6} - 3 \cdot \frac{1}{36} = \frac{1}{12}$. Since $u = \frac{t}{2^t}$ is continuous and attains the value $\frac{1}{6}$, the maximum is achieved.

Thus, the correct answer is **C**.

23. How many integers $n \geq 2$ are there such that whenever $z_1, z_2, \ldots, z_n$ are complex numbers such that

$$|z_1| = |z_2| = \cdots = |z_n| = 1 \quad \text{and} \quad z_1 + z_2 + \cdots + z_n = 0,$$

then the numbers $z_1, z_2, \ldots, z_n$ are equally spaced on the unit circle in the complex plane?

A 1

B 2

C 3

D 4

E 5

Solution:

For $n = 2$, $z_1 + z_2 = 0$ forces $z_2 = -z_1$, which is equally spaced. For $n = 3$, three unit vectors summing to zero must form an equilateral triangle, so they are equally spaced.

For every $n \geq 4$, a counterexample exists. For instance, take an antipodal pair $\{1, -1\}$ together with any other balanced set (for $n = 4$, use two antipodal pairs at different angles; for $n = 5$, use an equilateral triangle plus an antipodal pair). These sum to zero but are not equally spaced.

Hence only $n = 2$ and $n = 3$ work, giving 2 values.

Thus, the correct answer is **B**.

24. Let $D(n)$ denote the number of ways of writing the positive integer n as a product

$$n = f_1 \cdot f_2 \cdots f_k,$$

where $k \geq 1$, the f_i are integers strictly greater than 1, and the order in which the factors are listed matters (that is, two representations that differ only in the order of the factors are counted as distinct). For example, the number 6 can be written as $6, 2 \cdot 3$, and $3 \cdot 2$, so $D(6) = 3$. What is $D(96)$?

A 112

B 128

C 144

D 172

E 184

Solution:

The first factor f_1 can be any divisor $d > 1$, after which the rest is an ordered factorization of n/d. So $D(n) = \sum_{d|n,\, d>1} D(n/d)$, with $D(1) = 1$.

Computing over the divisors of $96 = 2^5 \cdot 3 : D(2) = D(3) = 1, D(4) = 2, D(6) = 3, D(8) = 4, D(12) = 8, D(16) = 8, D(24) = 20, D(32) = 16, D(48) = 48.$

Finally $D(96) = D(48) + D(32) + D(24) + D(16) + D(12) + D(8) + D(6) + D(4) + D(3) + D(2) + D(1) = 48 + 16 + 20 + 8 + 8 + 4 + 3 + 2 + 1 + 1 + 1 = 112.$

Thus, the correct answer is **A**.

25. For each real number a with $0 \leq a \leq 1$, let numbers x and y be chosen independently at random from the intervals $[0, a]$ and $[0, 1]$, respectively, and let $P(a)$ be the probability that

$$\sin^2(\pi x) + \sin^2(\pi y) > 1.$$

What is the maximum value of $P(a)$?

A $\dfrac{7}{12}$

B $2 - \sqrt{2}$

C $\dfrac{1+\sqrt{2}}{4}$

D $\dfrac{\sqrt{5}-1}{2}$

E $\dfrac{5}{8}$

Solution:

Since $\sin^2(\pi y) = 1 - \cos^2(\pi y)$, the condition is $|\sin \pi x| > |\cos \pi y|$. For fixed x, the probability over $y \in [0, 1]$ is $g(x) = 2x$ for $0 \leq x \leq \frac{1}{2}$ and $g(x) = 2 - 2x$ for $\frac{1}{2} \leq x \leq 1$.

Then $P(a) = \frac{1}{a} \int_0^a g(x)\, dx$. For $a \leq \frac{1}{2}$, $P(a) = a$, increasing to $\frac{1}{2}$. For $a \geq \frac{1}{2}$,

$$P(a) = \frac{2a - a^2 - \frac{1}{2}}{a} = 2 - a - \frac{1}{2a}.$$

Setting the derivative to zero gives $2a^2 = 1$, so $a = \frac{1}{\sqrt{2}}$, and $P\left(\frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2}$.

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc12/2020B