2020 AMC 12A Solutions

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1. Carlos took 70% of a whole pie. Maria took one third of the remainder. What portion of the whole pie was left?

A 10%

B 15%

[✓ C] 20%

D 30%

E 35%

Solution:

After Carlos takes 70%, the remaining portion is 30%.

Maria takes one third of this, namely $\frac{1}{3} \cdot 30\% = 10\%$, leaving $30\% - 10\% = 20\%$.

Thus, **C** is the correct answer.

2. The acronym AMC is shown in the rectangular grid below with grid lines spaced 1 unit apart. In units, what is the sum of the lengths of the line segments that form the acronym AMC?

A 17

B $15+2\sqrt{2}$

[✓ C] $13+4\sqrt{2}$

D $11+6\sqrt{2}$

E 21

Solution:

Split each letter into its segments. The A is a diagonal of length $2\sqrt{2}$, a vertical of length 2, and a crossbar of length 1.

The M has two verticals of length 2 and two diagonals of length $\sqrt{2}$ each. The C is three sides of length 2.

The straight pieces total $2+1+2+2+2+2+2=13$, and the diagonal pieces total $2\sqrt{2}+\sqrt{2}+\sqrt{2}=4\sqrt{2}$.

The sum is $13+4\sqrt{2}$.

Thus, **C** is the correct answer.

3. A driver travels for 2 hours at 60 miles per hour, during which her car gets 30 miles per gallon of gasoline. She is paid $\$0.50$ per mile, and her only expense is gasoline at $\$2.00$ per gallon. What is her net rate of pay, in dollars per hour, after this expense?

A 20

B 22

C 24

D 25

[✓ E] 26

Solution:

In 2 hours she drives 120 miles, earning $120 \cdot \$0.50 = \$60.$

She uses $120 \div 30 = 4$ gallons, costing $4 \cdot \$2.00 = \$8.$

Her net earnings are $\$60 - \$8 = \$52,$ so her rate is $\$52 \div 2 = \$26$ per hour.

Thus, **E** is the correct answer.

4. How many 4-digit positive integers (that is, integers between 1000 and 9999, inclusive) having only even digits are divisible by 5?

A 80

[✓ B] 100

C 125

D 200

E 500

Solution:

To be divisible by 5 the last digit is 0 or 5, and to be even it must be 0. So the units digit is fixed.

The leading digit is a nonzero even digit: $2, 4, 6, 8$ give 4 choices. Each of the two middle digits is any even digit $0, 2, 4, 6, 8$, giving 5 choices each.

The total is $4 \cdot 5 \cdot 5 \cdot 1 = 100.$

Thus, **B** is the correct answer.

5. The 25 integers from -10 to $14,$ inclusive, can be arranged to form a 5-by-5 square in which the sum of the numbers in each row, the sum of the numbers in each column, and the sum of the numbers along each of the main diagonals are all the same. What is the value of this common sum?

A 2

B 5

[✓ C] 10

D 25

E 50

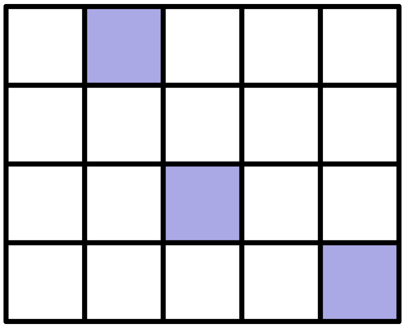
Solution:

The sum of the 25 integers is $\dfrac{(-10+14)\cdot 25}{2} = 50.$

The five rows each have the same sum and together account for the total, so each row sums to $50 \div 5 = 10.$

Thus, **C** is the correct answer.

6. In the plane figure shown below, 3 of the unit squares have been shaded. What is the least number of additional unit squares that must be shaded so that the resulting figure has two lines of symmetry?



A 4

B 5

C 6

[✓ D] 7

E 8

Solution:

For both symmetries, the lines must be the vertical and horizontal center lines of the 5-by-4 grid. Every shaded square then forces the squares obtained by reflecting it across each line.

The top square lies off-center, so its reflection group has 4 squares, requiring 3 more. The middle square sits on the central column, so its group has 2 squares, requiring 1 more. The bottom-right square again has a group of 4, requiring 3 more.

The least number of additional squares is $3 + 1 + 3 = 7.$

Thus, **D** is the correct answer.

7. Seven cubes, whose volumes are $1, 8, 27, 64, 125, 216,$ and 343 cubic units, are stacked vertically to form a tower in which the volumes of the cubes decrease from bottom to top. Except for the bottom cube, the bottom face of each cube lies completely on top of the cube below it. What is the total surface area of the tower (including the bottom) in square units?

A 644

[✓ B] 658

C 664

D 720

E 749

Solution:

The side lengths are $1, 2, \ldots, 7$. The four side faces of cube k contribute $4k^2$, so the vertical faces total $4(1^2 + 2^2 + \cdots + 7^2) = 4 \cdot 140 = 560$.

Viewed from directly above, every upward-facing horizontal patch projects onto the 7×7 base without overlap, giving 49. Viewed from below, the same is true, giving another 49.

The total surface area is $560 + 49 + 49 = 658$.

Thus, **B** is the correct answer.

8. What is the median of the following list of 4040 numbers?

$$1, 2, 3, \ldots, 2020, 1^2, 2^2, 3^2, \ldots, 2020^2$$

A 1974.5

B 1975.5

[✓ C] 1976.5

D 1977.5

E 1978.5

Solution:

The median is the average of the 2020th and 2021st smallest values.

The perfect squares that are at most 2020 are $1^2, \ldots, 44^2$ (since $44^2 = 1936$ and $45^2 = 2025$), so there are 44 of them.

Among the list, the numbers ≤ 1976 are the 1976 integers $1, \ldots, 1976$ together with those 44 squares, totaling $1976 + 44 = 2020.$

Thus the 2020th value is 1976 and the 2021st value is $1977,$ making the median

$$\frac{1976 + 1977}{2} = 1976.5.$$

Thus, **C** is the correct answer.

9. How many solutions does the equation $\tan(2x) = \cos\left(\frac{x}{2}\right)$ have on the interval $[0, 2\pi]$?

A 1

B 2

C 3

D 4

[✓ E] 5

Solution:

On $[0, 2\pi]$, the graph of $\cos\left(\frac{x}{2}\right)$ is a single arc decreasing from 1 down to -1.

The function $\tan(2x)$ has period $\frac{\pi}{2}$ with vertical asymptotes at $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. These split the interval into five stretches, and on each stretch $\tan(2x)$ runs through all real values.

Since the cosine arc is bounded, each of the five branches meets it exactly once, giving 5 solutions.

Thus, **E** is the correct answer.

10. There is a unique positive integer n such that

$$\log_2(\log_{16} n) = \log_4(\log_4 n).$$

What is the sum of the digits of n?

A 4

B 7

C 8

D 11

[✓ E] 13

Solution:

Since $\log_{16} n = \frac{1}{2}\log_4 n$, set $y = \log_4 n$. The equation becomes $\log_2\left(\frac{y}{2}\right) = \log_4 y = \frac{1}{2}\log_2 y$.

Multiplying by 2 gives $\log_2\left(\frac{y}{2}\right)^2 = \log_2 y$, so $\left(\frac{y}{2}\right)^2 = y$, which yields $y = 4$.

Then $\log_4 n = 4$, so $n = 4^4 = 256$, and the digit sum is $2 + 5 + 6 = 13$.

Thus, **E** is the correct answer.

11. A frog sitting at the point $(1, 2)$ begins a sequence of jumps, where each jump is parallel to one of the coordinate axes and has length 1, and the direction of each jump (up, down, right, or left) is chosen independently at random. The sequence ends when the frog reaches a side of the square with vertices $(0, 0)$, $(0, 4)$, $(4, 4)$, and $(4, 0)$. What is the probability that the sequence of jumps ends on a vertical side of the square?

A $\frac{1}{2}$

[✓ B] $\frac{5}{8}$

C $\frac{2}{3}$

D $\frac{3}{4}$

E $\frac{7}{8}$

Solution:

Let $P(x, y)$ be the probability of ending on a vertical side. On a vertical side $P = 1$, on a horizontal side $P = 0$, and at an interior point P is the average of its four neighbors.

By left-right symmetry $P(2, 2) = \frac{1}{2}$. Let $a = P(1, 2), b = P(1, 1) = P(1, 3)$, and $c = P(2, 1) = P(2, 3)$. Then

$a = \frac{1}{4}\left(1 + \frac{1}{2} + 2b\right)$, $b = \frac{1}{4}(1 + c + a)$, and $c = \frac{1}{4}\left(2b + \frac{1}{2}\right)$.

Substituting gives $b = \frac{1}{2}$, hence $a = \frac{3}{8} + \frac{1}{2}b = \frac{5}{8}$.

Thus, **B** is the correct answer.

12. Line ℓ in the coordinate plane has the equation $3x - 5y + 40 = 0$. This line is rotated 45° counterclockwise about the point $(20, 20)$ to obtain line k. What is the x-coordinate of the x-intercept of line k?

A 10

[✓ B] 15

C 20

D 25

E 30

Solution:

Note $(20, 20)$ satisfies $3x - 5y + 40 = 0$, so it is on ℓ and remains on k. The slope of ℓ is $\frac{3}{5}$.

Rotating by 45° gives slope $\dfrac{\frac{3}{5} + 1}{1 - \frac{3}{5}} = \dfrac{\frac{8}{5}}{\frac{2}{5}} = 4.$

Line k is $y - 20 = 4(x - 20)$. Setting $y = 0$ gives $-20 = 4(x - 20)$, so $x - 20 = -5$ and $x = 15$.

Thus, **B** is the correct answer.

13. There are integers $a, b,$ and c, each greater than 1, such that

$$\sqrt[a]{N\sqrt[b]{N\sqrt[c]{N}}} = \sqrt[36]{N^{25}}$$

for all $N > 1$. What is b?

A 2

[✓ B] 3

C 4

D 5

E 6

Solution:

The left side equals N raised to the exponent $\frac{1}{a} + \frac{1}{ab} + \frac{1}{abc}$, which must equal $\frac{25}{36}$.

Trying $a = 2$ leaves $\frac{1}{2b} + \frac{1}{2bc} = \frac{25}{36} - \frac{1}{2} = \frac{7}{36}$.

Then $\frac{1}{2b}\left(1 + \frac{1}{c}\right) = \frac{7}{36}$. Taking $b = 3$ gives $1 + \frac{1}{c} = \frac{7}{6}$, so $c = 6$.

All three are integers greater than 1, and $b = 3$.

Thus, **B** is the correct answer.

14. Regular octagon $ABCDEFGH$ has area n. Let m be the area of quadrilateral $ACEG$. What is $\frac{m}{n}$?

A $\frac{\sqrt{2}}{4}$

[✓ B] $\frac{\sqrt{2}}{2}$

C $\frac{3}{4}$

D $\frac{3\sqrt{2}}{5}$

E $\frac{2\sqrt{2}}{3}$

Solution:

The four vertices A, C, E, G form a square, since they are every other vertex of the regular octagon.

Taking a unit circumradius, the octagon's area is $2\sqrt{2}$ and the square $ACEG$ has diagonal equal to the circle's diameter, giving area 2.

The ratio is $\frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

Thus, **B** is the correct answer.

15. In the complex plane, let A be the set of solutions to $z^3 - 8 = 0$ and let B be the set of solutions to $z^3 - 8z^2 - 8z + 64 = 0$. What is the greatest distance between a point of A and a point of B?

A $2\sqrt{3}$

B 6

C 9

[✓ D] $2\sqrt{21}$

E $9 + \sqrt{3}$

Solution:

The set A consists of the cube roots of $8 : 2, -1 + i\sqrt{3},$ and $-1 - i\sqrt{3}.$

Factoring by grouping, $z^3 - 8z^2 - 8z + 64 = z^2(z - 8) - 8(z - 8) = (z - 8)(z^2 - 8),$ so $B = \{8, 2\sqrt{2}, -2\sqrt{2}\},$ all real.

The greatest distance is from $-1 \pm i\sqrt{3}$ to $8 : \sqrt{(8 - (-1))^2 + (\sqrt{3})^2} = \sqrt{81 + 3} = \sqrt{84} = 2\sqrt{21}.$

Thus, **D** is the correct answer.

16. A point is chosen at random within the square in the coordinate plane whose vertices are $(0, 0)$, $(2020, 0)$, $(2020, 2020)$, and $(0, 2020)$. The probability that the point is within d units of a lattice point is $\frac{1}{2}$. (A point (x, y) is a lattice point if x and y are both integers.) What is d to the nearest tenth?

A 0.3

[✓ B] 0.4

C 0.5

D 0.6

E 0.7

Solution:

By periodicity it suffices to consider one unit cell with a lattice point at each corner. The region within d of a corner consists of four quarter-disks of radius d, forming one full disk of area πd^2.

Setting $\pi d^2 = \frac{1}{2}$ gives $d = \sqrt{\dfrac{1}{2\pi}} \approx 0.399.$

To the nearest tenth, $d = 0.4.$

Thus, **B** is the correct answer.

17. The vertices of a quadrilateral lie on the graph of $y = \ln x$, and the x-coordinates of these vertices are consecutive positive integers. The area of the quadrilateral is $\ln \frac{91}{90}$. What is the x-coordinate of the leftmost vertex?

A 6

B 7

C 10

[✓ D] 12

E 13

Solution:

Let the vertices have x-coordinates $n, n+1, n+2, n+3$ with y-coordinates $\ln$ of those values. Applying the shoelace formula and simplifying, the area is $\ln \dfrac{(n+1)(n+2)}{n(n+3)}$.

Setting $\dfrac{(n+1)(n+2)}{n(n+3)} = \dfrac{91}{90}$ gives $1 + \dfrac{2}{n^2+3n} = \dfrac{91}{90}$, so $n^2 + 3n = 180.$

Then $(n-12)(n+15) = 0$, so $n = 12.$

Thus, **D** is the correct answer.

18. Quadrilateral $ABCD$ satisfies $\angle ABC = \angle ACD = 90^\circ$, $AC = 20$, and $CD = 30$. Diagonals AC and BD intersect at point E, and $AE = 5$. What is the area of quadrilateral $ABCD$?

A 330

B 340

C 350

[✓ D] 360

E 370

Solution:

Place $A = (0, 0)$ and $C = (20, 0)$. Since $\angle ACD = 90^\circ$, $D = (20, 30)$, and $E = (5, 0)$ because $AE = 5$.

Since $\angle ABC = 90^\circ$, B lies on the circle of radius 10 centered at $(10, 0)$. Line DE is $(5 + t,\ 2t)$; substituting gives $t^2 - 2t - 15 = 0$, so $t = 5$ or $t = -3$.

For E to lie between B and D, take $t = -3$, giving $B = (2, -6)$, a distance 6 below line AC.

Then $[ACD] = \frac{1}{2} \cdot 20 \cdot 30 = 300$ and $[ABC] = \frac{1}{2} \cdot 20 \cdot 6 = 60$, so the total area is 360.

Thus, **D** is the correct answer.

19. There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \cdots < a_k$ such that

$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \cdots + 2^{a_k}.$$

What is k?

A 117

B 136

[✓ C] 137

D 273

E 306

Solution:

Let $x = 2^{17}$. Then $\dfrac{2^{289}+1}{2^{17}+1} = \dfrac{x^{17}+1}{x+1} = x^{16} - x^{15} + \cdots - x + 1$, an alternating sum of the 17 powers $x^0, x^1, \ldots, x^{16}$.

Pair each subtracted power with the added power just above it: $x^{m+1} - x^m = 2^{17m}(2^{17}-1) = 2^{17m} + 2^{17m+1} + \cdots + 2^{17m+16}$, a block of 17 consecutive powers of 2.

There are 8 such pairs, together with the leftover $+2^0$. The blocks occupy disjoint ranges, so the total number of powers is $8 \cdot 17 + 1 = 137$.

Thus, **C** is the correct answer.

20. Let T be the triangle in the coordinate plane with vertices $(0, 0)$, $(4, 0)$, and $(0, 3)$. Consider the following five isometries (rigid transformations) of the plane: rotations of 90°, 180°, and 270° counterclockwise around the origin, reflection across the x-axis, and reflection across the y-axis. How many of the 125 sequences of three of these transformations (not necessarily distinct) will return T to its original position? (For example, a 180° rotation, followed by a reflection across the x-axis, followed by a reflection across the y-axis will return T to its original position, but a 90° rotation, followed by a reflection across the x-axis, followed by another reflection across the x-axis will not return T to its original position.)

[✓ A] 12

B 15

C 17

D 20

E 25

Solution:

Because T is a scalene right triangle, the only isometry carrying T to itself is the identity, so a sequence works exactly when the three transformations compose to the identity.

The five maps are all of the square's symmetry group except the identity and the two diagonal reflections. In an ordered triple the third map must be the inverse of the first two composed, and it is allowed precisely when the product of the first two is again one of the five.

Of the 25 ordered pairs, 5 compose to the identity and 8 compose to a diagonal reflection. The remaining $25 - 13 = 12$ pairs give valid sequences.

Thus, **A** is the correct answer.

21. How many positive integers n are there such that n is a multiple of 5, and the least common multiple of $5!$ and n equals 5 times the greatest common divisor of $10!$ and n?

A 12

B 24

C 36

[✓ D] 48

E 72

Solution:

Write $n = 2^a 3^b 5^c 7^d \cdots$. Since $5! = 2^3 \cdot 3 \cdot 5$ has no other primes, n can only involve $2, 3, 5, 7$. Matching exponents in $\operatorname{lcm}(5!, n) = 5 \cdot \gcd(10!, n)$:

For $2 : \max(3, a) = \min(8, a)$, so $3 \leq a \leq 8$ gives 6 values. For $3 : \max(1, b) = \min(4, b)$, so $1 \leq b \leq 4$ gives 4 values.

For $5 : \max(1, c) = 1 + \min(2, c)$ with $c \geq 1$, which forces $c = 3$, giving 1 value. For $7 : \max(0, d) = \min(1, d)$, so $d = 0$ or 1, giving 2 values.

The total is $6 \cdot 4 \cdot 1 \cdot 2 = 48$.

Thus, **D** is the correct answer.

22. Let (a_n) and (b_n) be the sequences of real numbers such that

$$(2+i)^n = a_n + b_n i$$

for all integers $n \geq 0$, where $i = \sqrt{-1}$. What is

$$\sum_{n=0}^{\infty} \frac{a_n b_n}{7^n}?$$

A $\frac{3}{8}$

[✓ B] $\frac{7}{16}$

C $\frac{1}{2}$

D $\frac{9}{16}$

E $\frac{4}{7}$

Solution:

Since $(a_n + b_n i)^2 = a_n^2 - b_n^2 + 2a_n b_n i$, we have $a_n b_n = \frac{1}{2}\operatorname{Im}\left((2+i)^{2n}\right) = \frac{1}{2}\operatorname{Im}\left((3+4i)^n\right)$.

Therefore the sum is $\frac{1}{2}\operatorname{Im}\sum_{n=0}^{\infty}\left(\frac{3+4i}{7}\right)^n = \frac{1}{2}\operatorname{Im}\left(\frac{1}{1-\frac{3+4i}{7}}\right)$.

This equals $\frac{1}{2}\operatorname{Im}\left(\frac{7}{4-4i}\right) = \frac{1}{2}\operatorname{Im}\left(\frac{7(4+4i)}{32}\right) = \frac{1}{2}\cdot\frac{28}{32} = \frac{7}{16}$.

Thus, **B** is the correct answer.

23. Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

[✓ A] $\frac{7}{36}$

B $\frac{5}{24}$

C $\frac{2}{9}$

D $\frac{17}{72}$

E $\frac{1}{4}$

Solution:

Rerolling one die, keeping two dice that sum to s, wins with probability $\frac{1}{6}$ when $s \leq 6$ and 0 otherwise. Rerolling two dice, keeping a die of value v, wins with probability equal to the number of ways two dice sum to $7 - v$, over 36; this is largest when v is smallest. Rerolling all three wins with probability $\frac{15}{216}$.

Rerolling exactly two dice is strictly best precisely when the two smallest dice sum to at least 7 (so rerolling one cannot reach 7) while the smallest die is $1, 2,$ or 3 (so keeping it beats rerolling all three).

Counting the ordered rolls with this property gives 42 out of 216, a probability of $\dfrac{42}{216} = \dfrac{7}{36}$.

Thus, **A** is the correct answer.

24. Suppose that $\triangle ABC$ is an equilateral triangle of side length s, with the property that there is a unique point P inside the triangle such that $AP = 1, BP = \sqrt{3}$, and $CP = 2$. What is s?

A $1+\sqrt{2}$

[✓ B] $\sqrt{7}$

C $\frac{8}{3}$

D $\sqrt{5+\sqrt{5}}$

E $2\sqrt{2}$

Solution:

A point at distances p, q, r from the vertices of an equilateral triangle of side s satisfies $3(p^4+q^4+r^4+s^4) = (p^2+q^2+r^2+s^2)^2$.

With $p^2 = 1, q^2 = 3, r^2 = 4$, letting $S = s^2$ gives $3(26+S^2) = (8+S)^2$, so $S^2 - 8S + 7 = 0$ and $S = 1$ or $S = 7$.

A triangle of side 1 cannot contain a point at distance 2 from a vertex, so $S = 7$ and $s = \sqrt{7}$.

Thus, **B** is the correct answer.

25. The number $a = \dfrac{p}{q}$, where p and q are relatively prime positive integers, has the property that the sum of all real numbers x satisfying

$$\lfloor x \rfloor \cdot \{x\} = a \cdot x^2$$

is 420, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x. What is $p + q$?

A 245

B 593

[✓ C] 929

D 1331

E 1332

Solution:

On the interval $x \in [n, n+1)$ the equation becomes $n(x - n) = ax^2$, i.e. $ax^2 - nx + n^2 = 0$, whose two roots are $x = \dfrac{n\left(1 \pm \sqrt{1 - 4a}\right)}{2a}$.

For $0 < a < \frac{1}{4}$ each interval $[n, n+1)$ contributes exactly one root of this quadratic that lies in it (for suitable n), and summing the roots over all valid n gives a total that depends only on a.

Requiring the sum to be 420 forces $a = \dfrac{29}{900}$, which is already in lowest terms. Hence $p + q = 29 + 900 = 929.$

Thus, **C** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc12/2020A