2018 AMC 12B Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc12/2018B/solutions

1. Kate bakes a 20-inch by 18-inch pan of cornbread. The cornbread is cut into pieces that measure 2 inches by 2 inches. How many pieces of cornbread does the pan contain?

A 90

B 100

C 180

D 200

E 360

Solution:

The pan has area $20 \cdot 18 = 360$ square inches, and each piece has area $2 \cdot 2 = 4$ square inches.

The number of pieces is

$$\frac{360}{4} = 90.$$

Thus, the correct answer is **A**.

2. Sam drove 96 miles in 90 minutes. His average speed during the first 30 minutes was 60 mph (miles per hour), and his average speed during the second 30 minutes was 65 mph. What was his average speed, in mph, during the last 30 minutes?

- A 64
- B 65
- C 66
- D 67
- E 68

Solution:

In the first 30 minutes Sam covered $60 \cdot \frac{1}{2} = 30$ miles, and in the second he covered $65 \cdot \frac{1}{2} = 32.5$ miles.

The last 30 minutes covered $96 - 30 - 32.5 = 33.5$ miles, so the speed was

$$\frac{33.5}{1/2} = 67 \text{ mph.}$$

Thus, the correct answer is **D**.

3. A line with slope 2 intersects a line with slope 6 at the point $(40, 30)$. What is the distance between the x-intercepts of these two lines?

- A 5
- B 10
- C 20
- D 25
- E 50

Solution:

The line of slope 2 is $y - 30 = 2(x - 40)$; setting $y = 0$ gives $x = 25$. The line of slope 6 is $y - 30 = 6(x - 40)$; setting $y = 0$ gives $x = 35$.

The distance between the intercepts is $|35 - 25| = 10$.

Thus, the correct answer is **B**.

4. A circle has a chord of length 10, and the distance from the center of the circle to the chord is 5. What is the area of the circle?

A 25π

B 50π

C 75π

D 100π

E 125π

Solution:

Dropping a perpendicular from the center to the chord bisects it, forming a right triangle with legs 5 (half the chord) and 5 (the distance), and hypotenuse r.

Then $r^2 = 5^2 + 5^2 = 50$, so the area is $\pi r^2 = 50\pi$.

Thus, the correct answer is **B**.

5. How many subsets of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?

A 128

B 192

C 224

D 240

E 256

Solution:

The set has 8 elements, giving $2^8 = 256$ subsets. The subsets with no prime use only the four non-primes $\{4, 6, 8, 9\}$, and there are $2^4 = 16$ of these.

So the number containing at least one prime is $256 - 16 = 240.$

Thus, the correct answer is **D**.

6. Suppose S cans of soda can be purchased from a vending machine for Q quarters. Which of the following expressions describes the number of cans of soda that can be purchased for D dollars, where 1 dollar is worth 4 quarters?

A $\frac{4DQ}{S}$

B $\frac{4DS}{Q}$

C $\frac{4Q}{DS}$

D $\frac{DQ}{4S}$

E $\frac{DS}{4Q}$

Solution:

One can costs $\frac{Q}{S}$ quarters, which is $\frac{Q}{4S}$ dollars. The number of cans that D dollars can buy is

$$\frac{D}{\frac{Q}{4S}} = \frac{4DS}{Q}.$$

Thus, the correct answer is **B**.

7. What is the value of

$$\log_3 7 \cdot \log_5 9 \cdot \log_7 11 \cdot \log_9 13 \cdots \log_{21} 25 \cdot \log_{23} 27?$$

- A 3
- B $3\log_7 23$
- C 6
- D 9
- E 10

Solution:

The factors split into two telescoping chains. The odd-position factors form

$$\log_3 7 \cdot \log_7 11 \cdot \log_{11} 15 \cdots \log_{23} 27 = \log_3 27 = 3,$$

and the even-position factors form

$$\log_5 9 \cdot \log_9 13 \cdots \log_{21} 25 = \log_5 25 = 2.$$

The product is $3 \cdot 2 = 6.$

Thus, the correct answer is **C**.

8. Line segment $\overline{AB}$ is a diameter of a circle with $AB = 24$. Point C, not equal to A or B, lies on the circle. As point C moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?

- A 25
- B 38
- C 50
- D 63
- E 75

Solution:

Let O be the center of the circle. The centroid of $\triangle ABC$ is the average of A, B, and C; since O is the midpoint of $\overline{AB}$, the centroid lies one-third of the way from O to C.

As C traces the circle of radius 12, the centroid traces a circle of radius $\frac{1}{3} \cdot 12 = 4$. Its area is $16\pi \approx 50$.

Thus, the correct answer is **C**.

9. What is

$$\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j)?$$

- A $100{,}100$
- B $500{,}500$
- C $505{,}000$
- D $1{,}001{,}000$
- E $1{,}010{,}000$

Solution:

Splitting the sum,

$$\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j) = \sum_{i=1}^{100}\sum_{j=1}^{100} i + \sum_{i=1}^{100}\sum_{j=1}^{100} j = 100\sum_{i=1}^{100} i + 100\sum_{j=1}^{100} j.$$

Since $\sum_{k=1}^{100} k = 5050$, this equals $100 \cdot 5050 + 100 \cdot 5050 = 1{,}010{,}000$.

Thus, the correct answer is **E**.

10. A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

A 202

B 223

C 224

D 225

E 234

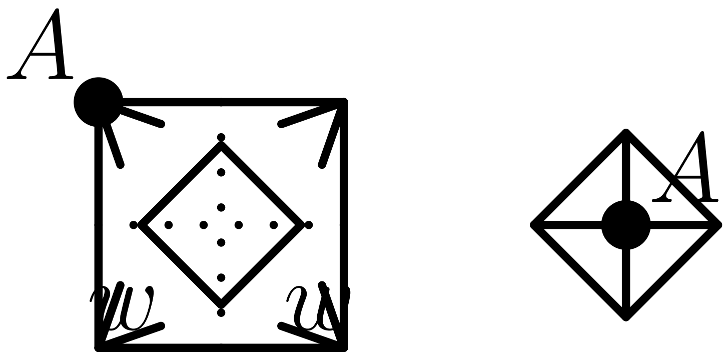
Solution:

The mode uses 10 of the entries, leaving $2008.$ Because the mode is unique, every other value appears at most 9 times, so at least $\left\lceil \frac{2008}{9} \right\rceil = 224$ distinct non-mode values are needed.

Adding the mode gives $224 + 1 = 225.$ This is achievable: use 9 copies each of 1 through $223,$ ten copies of $224,$ and one copy of $225.$

Thus, the correct answer is **D**.

11. A closed box with a square base is to be wrapped with a square sheet of wrapping paper. The box is centered on the wrapping paper with the vertices of the base lying on the midlines of the square sheet of paper, as shown in the figure on the left. The four corners of the wrapping paper are to be folded up over the sides and brought together to meet at the center of the top of the box, point A in the figure on the right. The box has base length w and height h. What is the area of the sheet of wrapping paper?



A $2(w+h)^2$

B $\dfrac{(w+h)^2}{2}$

C $2w^2+4wh$

D $2w^2$

E w^2h

Solution:

Following a fold from a corner of the paper to the center of the box top, the distance from a corner of the sheet to its center is

$$\frac{w}{2}+h+\frac{w}{2}=w+h.$$

That segment is a leg of a 45-45-90 triangle whose hypotenuse is a full side of the square sheet, so the side length is $\sqrt{2}\,(w+h)$.

The area of the sheet is $\left(\sqrt{2}\,(w+h)\right)^2=2(w+h)^2$.

Thus, the correct answer is **A**.

12. Side $\overline{AB}$ of $\triangle ABC$ has length $10.$ The bisector of angle A meets $\overline{BC}$ at $D,$ and $CD = 3.$ The set of all possible values of AC is an open interval $(m, n).$ What is $m + n?$

A 16

B 17

C 18

D 19

E 20

Solution:

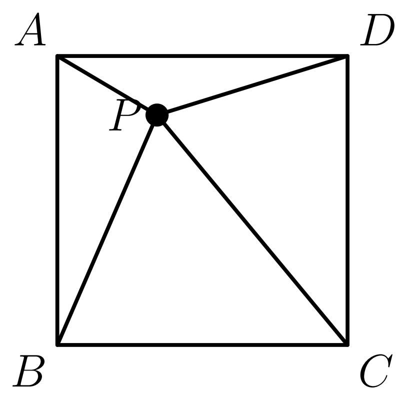
Let $q = AC$ and $r = BD.$ The angle bisector theorem gives $\frac{q}{3} = \frac{10}{r},$ so $r = \frac{30}{q}.$

Applying the triangle inequalities to sides $q, 10,$ and $3 + r$ and substituting $r = \frac{30}{q}$ yields $(q - 15)(q + 2) < 0$ and $(q - 3)(q + 10) > 0$ (the third inequality holds automatically). Together these force $3 < q < 15.$

So $(m, n) = (3, 15)$ and $m + n = 18.$

Thus, the correct answer is **C**.

13. Square $ABCD$ has side length 30. Point P lies inside the square so that $AP = 12$ and $BP = 26$. The centroids of $\triangle ABP, \triangle BCP, \triangle CDP,$ and $\triangle DAP$ are the vertices of a convex quadrilateral. What is the area of that quadrilateral?



A $100\sqrt{2}$

B $100\sqrt{3}$

C 200

D $200\sqrt{2}$

E $200\sqrt{3}$

Solution:

Place $A = (0, 30), B = (0, 0), C = (30, 0), D = (30, 30)$, and $P = (3x, 3y)$. Averaging the vertices, the four centroids are

$$(x,\ y + 10),\ (x + 10,\ y),\ (x + 20,\ y + 10),\ (x + 10,\ y + 20).$$

These form a square whose diagonals, one horizontal and one vertical, each have length 20. Its area is $\frac{1}{2} \cdot 20 \cdot 20 = 200$, independent of where P lies.

Thus, the correct answer is **C**.

14. Joey and Chloe and their daughter Zoe all have the same birthday. Joey is 1 year older than Chloe, and Zoe is exactly 1 year old today. Today is the first of the 9 birthdays on which Chloe's age will be an integral multiple of Zoe's age. What will be the sum of the two digits of Joey's age the next time his age is a multiple of Zoe's age?

A 7

B 8

C 9

D 10

E 11

Solution:

Let Chloe be n today, so she is $n-1$ years older than Zoe. In y years Chloe's age $n+y$ is a multiple of Zoe's age $1+y$ exactly when $1+y$ divides $n-1$. Having 9 such birthdays means $n-1$ has exactly 9 divisors.

A number with exactly 9 divisors has the form p^2q^2 or p^8; the only two-digit case is $2^2 \cdot 3^2 = 36$. So Chloe is 37 and Joey is 38.

Joey's age $38+y$ is a multiple of $1+y$ exactly when $1+y$ divides 37. The next time is $y=36$, making Joey 74, with digit sum $7+4=11$.

Thus, the correct answer is **E**.

15. How many 3-digit positive odd multiples of 3 do not include the digit 3?

- A 96
- B 97
- C 98
- D 102
- E 120

Solution:

Write the number as $\overline{abc}$. The hundreds digit a has 8 choices $(1, 2, 4, 5, 6, 7, 8, 9)$, and the units digit c has 4 choices $(1, 5, 7, 9)$.

The tens digit b may be any of $\{0, 1, 2, 4, 5, 6, 7, 8, 9\}$. These split into three residue classes mod 3 of equal size $\{0, 6, 9\}, \{1, 4, 7\}, \{2, 5, 8\}$, so exactly 3 choices of b make $a + b + c$ divisible by 3.

The count is $8 \cdot 4 \cdot 3 = 96$.

Thus, the correct answer is **A**.

16. The solutions to the equation $(z+6)^8 = 81$ are connected in the complex plane to form a convex regular polygon, three of whose vertices are labeled $A, B,$ and C. What is the least possible area of $\triangle ABC$?

A $\frac{1}{6}\sqrt{6}$

B $\frac{3}{2}\sqrt{2} - \frac{3}{2}$

C $2\sqrt{3} - 2\sqrt{2}$

D $\frac{1}{2}\sqrt{2}$

E $\sqrt{3} - 1$

Solution:

Translating by 6, the solutions of $z^8 = 81$ are eight points on a circle of radius $81^{1/8} = \sqrt{3}$, forming a regular octagon. The minimum-area triangle uses three consecutive vertices.

Take $A = \left(\frac{1}{2}\sqrt{6}, \frac{1}{2}\sqrt{6}\right)$, $B = (\sqrt{3}, 0)$, and $C = \left(\frac{1}{2}\sqrt{6}, -\frac{1}{2}\sqrt{6}\right)$. Then $AC = \sqrt{6}$ and the height is $\sqrt{3} - \frac{1}{2}\sqrt{6}$, so the area is

$$\frac{1}{2} \cdot \sqrt{6}\left(\sqrt{3} - \frac{1}{2}\sqrt{6}\right) = \frac{3}{2}\sqrt{2} - \frac{3}{2}.$$

Thus, the correct answer is **B**.

17. Let p and q be positive integers such that

$$\frac{5}{9} < \frac{p}{q} < \frac{4}{7}$$

and q is as small as possible. What is $q - p$?

A 7

B 11

C 13

D 17

E 19

Solution:

From $\frac{5}{9} < \frac{p}{q}$ we get $9p - 5q \geq 1$, and from $\frac{p}{q} < \frac{4}{7}$ we get $4q - 7p \geq 1$. Now

$$\frac{1}{63} = \frac{4}{7} - \frac{5}{9} = \frac{4q - 7p}{7q} + \frac{9p - 5q}{9q} \geq \frac{1}{7q} + \frac{1}{9q} = \frac{16}{63q}.$$

Hence $q \geq 16$. With $q = 16$, the fraction $\frac{9}{16}$ lies strictly between $\frac{5}{9}$ and $\frac{4}{7}$, so $p = 9$ and $q - p = 16 - 9 = 7$.

Thus, the correct answer is **A**.

18. A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$?

A 2016

B 2017

C 2018

D 2019

E 2020

Solution:

Repeatedly substituting the recursion into itself gives

$$f(n) = f(n-6) + 6.$$

So f increases by 6 every time n increases by 6.

Since $2018 = 2 + 6 \cdot 336$, we have $f(2018) = f(2) + 6 \cdot 336 = 1 + 2016 = 2017.$

Thus, the correct answer is **B**.

19. Mary chose an even 4-digit number n. She wrote down all the divisors of n in increasing order from left to right: $1, 2, \ldots, \frac{n}{2}, n$. At some moment Mary wrote 323 as a divisor of n. What is the smallest possible value of the next divisor written to the right of 323?

A 324

B 330

C 340

D 361

E 646

Solution:

Let d be the next divisor after 323. If $\gcd(d, 323) = 1$, then $323d$ divides n, forcing $n \geq 323d > 323^2 > 9999$, impossible for a 4-digit number. So d shares a prime factor with $323 = 17 \cdot 19$.

Then $d - 323 \geq \gcd(d, 323) \geq 17$, so $d \geq 340$. Indeed $d = 340 = 17 \cdot 20$ occurs for $n = 17 \cdot 19 \cdot 20 = 6460$, which is even and 4-digit.

Thus, the correct answer is **C**.

20. Let $ABCDEF$ be a regular hexagon with side length 1. Denote by X, Y, and Z the midpoints of sides AB, CD, and EF, respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$?

A $\frac{3}{8}\sqrt{3}$

B $\frac{7}{16}\sqrt{3}$

C $\frac{15}{32}\sqrt{3}$

D $\frac{1}{2}\sqrt{3}$

E $\frac{9}{16}\sqrt{3}$

Solution:

Both $\triangle ACE$ and $\triangle XYZ$ are equilateral, and $\triangle ACE$ has half the area of the hexagon. The vertices where the two triangles cut each other let the shaded hexagon be measured against the midpoint triangle U of $\triangle ACE$.

That midpoint triangle has $\frac{1}{4}$ the area of $\triangle ACE$, hence $\frac{1}{8}$ of the hexagon. The shaded region equals $\frac{5}{2}$ of the midpoint triangle, so it is $\frac{5}{2} \cdot \frac{1}{8} = \frac{5}{16}$ of the hexagon.

The hexagon has area $6 \cdot \frac{\sqrt{3}}{4} \cdot 1^2 = \frac{3\sqrt{3}}{2}$, so the shaded area is $\frac{5}{16} \cdot \frac{3\sqrt{3}}{2} = \frac{15}{32}\sqrt{3}.$

Thus, the correct answer is **C**.

21. In $\triangle ABC$ with side lengths $AB = 13, AC = 12$, and $BC = 5$, let O and I denote the circumcenter and incenter, respectively. A circle with center M is tangent to the legs AC and BC and to the circumcircle of $\triangle ABC$. What is the area of $\triangle MOI$?

A $\frac{5}{2}$

B $\frac{11}{4}$

C 3

D $\frac{13}{4}$

E $\frac{7}{2}$

Solution:

Since $5^2 + 12^2 = 13^2$, the triangle is right-angled at C. Set $C = (0,0), A = (12,0)$, and $B = (0,5)$. Then O is the midpoint of $\overline{AB}$, namely $O = \left(6, \frac{5}{2}\right)$, with circumradius $\frac{13}{2}$. The inradius is $\frac{\text{area}}{s} = \frac{30}{15} = 2$, so $I = (2,2)$.

Because M's circle is tangent to both legs, $M = (\rho, \rho)$. Internal tangency to the circumcircle gives $MO = \frac{13}{2} - \rho$. Setting this equal to $\sqrt{(\rho-6)^2 + \left(\rho - \frac{5}{2}\right)^2}$ and solving gives $\rho = 4$, so $M = (4,4)$.

The shoelace formula on $M = (4,4), O = \left(6, \frac{5}{2}\right), I = (2,2)$ gives area $\frac{7}{2}$.

Thus, the correct answer is **E**.

22. Consider polynomials $P(x)$ of degree at most 3, each of whose coefficients is an element of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many such polynomials satisfy $P(-1) = -9$?

A 110

B 143

C 165

D 220

E 286

Solution:

Write $P(x) = ax^3 + bx^2 + cx + d$ with each of a, b, c, d in $\{0, \ldots, 9\}$. The condition is $-a + b - c + d = -9$.

Let $a' = 9 - a$ and $c' = 9 - c$, both in $[0, 9]$. Then $a' + b + c' + d = 9$. By stars and bars the number of nonnegative solutions is $\binom{9+3}{3} = \binom{12}{3} = 220$, and each automatically satisfies the upper bounds since the sum is 9.

Thus, the correct answer is **D**.

23. Ajay is standing at point A near Pontianak, Indonesia, 0° latitude and 110° E longitude. Billy is standing at point B near Big Baldy Mountain, Idaho, USA, 45° N latitude and 115° W longitude. Assume that Earth is a perfect sphere with center C. What is the degree measure of $\angle ACB$?

A 105

B $112\frac{1}{2}$

C 120

D 135

E 150

Solution:

The longitudes differ by $360^\circ - (110^\circ + 115^\circ) = 135^\circ$, and B is at latitude 45° N. Place $A = (1, 0, 0)$ on the unit sphere.

Then $B = (\cos 45^\circ \cos 135^\circ,\ \cos 45^\circ \sin 135^\circ,\ \sin 45^\circ) = \left(-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$. The dot product is $A \cdot B = -\frac{1}{2}$, so $\cos \angle ACB = -\frac{1}{2}$ and $\angle ACB = 120^\circ$.

Thus, the correct answer is **C**.

24. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. How many real numbers x satisfy the equation $x^2 + 10{,}000\lfloor x \rfloor = 10{,}000x$?

- A 197
- B 198
- C 199
- D 200
- E 201

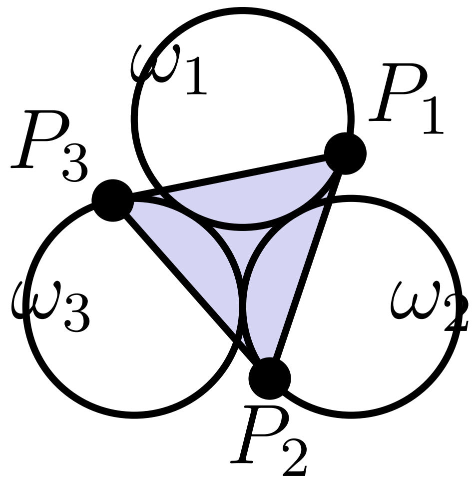
Solution:

Let $\{x\} = x - \lfloor x \rfloor$. The equation becomes $x^2 = 10{,}000\{x\}$, so $\frac{x^2}{10{,}000} = \{x\}$. Since $0 \le \{x\} < 1$, we need $0 \le x^2 < 10{,}000$, i.e. $-100 < x < 100$.

On each interval $[k, k+1)$ the increasing parabola $\frac{x^2}{10{,}000}$ meets the segment $\{x\}$ exactly once. These intervals run for $k = -100, -99, \ldots, 98$, giving 199 solutions.

Thus, the correct answer is **C**.

25. Circles $\omega_1, \omega_2,$ and ω_3 each have radius 4 and are placed in the plane so that each circle is externally tangent to the other two. Points $P_1, P_2,$ and P_3 lie on $\omega_1, \omega_2,$ and $\omega_3,$ respectively, so that $P_1P_2 = P_2P_3 = P_3P_1$ and line P_iP_{i+1} is tangent to ω_i for each $i = 1, 2, 3,$ where $P_4 = P_1$. See the figure below. The area of $\triangle P_1P_2P_3$ can be written in the form $\sqrt{a} + \sqrt{b}$, where a and b are positive integers. What is $a + b$?



A 546

B 548

C 550

D 552

E 554

Solution:

Let O_i be the center of ω_i, and let K be the intersection of lines O_1P_1 and O_2P_2. Because $\angle P_1P_2P_3 = 60^\circ$, triangle P_2KP_1 is a 30-60-90° triangle. With $d = P_1K$, we get $P_2K = 2d$ and $P_1P_2 = \sqrt{3}\,d$.

The Law of Cosines in $\triangle O_1KO_2$ (with $O_1O_2 = 8$) gives

$$8^2 = (d+4)^2 + (2d-4)^2 - 2(d+4)(2d-4)\cos 60^\circ,$$

which simplifies to $3d^2 - 12d - 16 = 0$, so $d = 2 + \frac{2}{3}\sqrt{21}$.

Then $P_1P_2 = \sqrt{3}\,d = 2\sqrt{3} + 2\sqrt{7}$, and the area is

$$\frac{\sqrt{3}}{4}\left(2\sqrt{3} + 2\sqrt{7}\right)^2 = 10\sqrt{3} + 6\sqrt{7} = \sqrt{300} + \sqrt{252}.$$

So $a + b = 300 + 252 = 552.$

Thus, the correct answer is **D**.

Problems: https://live.poshenloh.com/past-contests/amc12/2018B