2017 AMC 12B Solutions

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1. Kymbrea's comic book collection currently has 30 comic books in it, and she is adding to her collection at the rate of 2 comic books per month. LaShawn's collection currently has 10 comic books in it, and he is adding to his collection at the rate of 6 comic books per month. After how many months will LaShawn's collection have twice as many comic books as Kymbrea's?

A 1

B 4

C 5

D 20

E 25

Solution:

After m months, Kymbrea has $30 + 2m$ comic books and LaShawn has $10 + 6m$. Setting $10 + 6m = 2(30 + 2m)$ gives $10 + 6m = 60 + 4m$, so $2m = 50$ and $m = 25$.

Thus, the correct answer is **E**.

2. Real numbers $x, y,$ and z satisfy the inequalities

$$0 < x < 1, \quad -1 < y < 0, \quad \text{and} \quad 1 < z < 2.$$

Which of the following numbers is necessarily positive?

A $y + x^2$

B $y + xz$

C $y + y^2$

D $y + 2y^2$

E $y + z$

Solution:

Adding $y > -1$ and $z > 1$ gives $y + z > 0$, so $y + z$ is always positive. Each of the other four choices can be made negative: with $x = \frac{1}{8}, y = -\frac{1}{4}, z = \frac{3}{2}$, every one of $y + x^2, y + xz, y + y^2,$ and $y + 2y^2$ is negative.

Thus, the correct answer is **E**.

3. Suppose that x and y are nonzero real numbers such that

$$\frac{3x+y}{x-3y} = -2.$$

What is the value of

$$\frac{x+3y}{3x-y}?$$

- A -3
- B -1
- C 1
- D 2
- E 3

Solution:

The equation gives $3x+y = -2(x-3y) = -2x+6y$, so $5x = 5y$, meaning $x = y$. Then

$$\frac{x+3y}{3x-y} = \frac{y+3y}{3y-y} = \frac{4y}{2y} = 2.$$

Thus, the correct answer is **D**.

4. Samia set off on her bicycle to visit her friend, traveling at an average speed of 17 kilometers per hour. When she had gone half the distance to her friend's house, a tire went flat, and she walked the rest of the way at 5 kilometers per hour. In all it took her 44 minutes to reach her friend's house. In kilometers rounded to the nearest tenth, how far did Samia walk?

- A 2.0
- B 2.2
- C 2.8
- D 3.4
- E 4.4

Solution:

Let $2d$ be the total distance, so she biked d at 17 km/h and walked d at 5 km/h. The total time in hours is

$$\frac{d}{17}+\frac{d}{5}=\frac{44}{60}.$$

Combining the left side gives $\frac{22d}{85}=\frac{11}{15}$, so $d=\frac{11}{15}\cdot\frac{85}{22}=\frac{17}{6}=2.833\ldots$ She walked about 2.8 kilometers.

Thus, the correct answer is **C**.

5. The data set $[6, 19, 33, 33, 39, 41, 41, 43, 51, 57]$ has median $Q_2 = 40$, first quartile $Q_1 = 33$, and third quartile $Q_3 = 43$. An outlier in a data set is a value that is more than 1.5 times the interquartile range below the first quartile (Q_1) or more than 1.5 times the interquartile range above the third quartile (Q_3), where the interquartile range is defined as $Q_3 - Q_1$. How many outliers does this data set have?

- A 0
- B 1
- C 2
- D 3
- E 4

Solution:

The interquartile range is $43 - 33 = 10$, so 1.5 times it is 15. Outliers are values less than $33 - 15 = 18$ or greater than $43 + 15 = 58$. Only 6 falls below 18, and nothing exceeds 58, so there is exactly 1 outlier.

Thus, the correct answer is **B**.

6. The circle having $(0, 0)$ and $(8, 6)$ as the endpoints of a diameter intersects the x-axis at a second point. What is the x-coordinate of this point?

A $4\sqrt{2}$

B 6

C $5\sqrt{2}$

D 8

E $6\sqrt{2}$

Solution:

The center is the midpoint of the diameter, $(4, 3)$, and the radius is $\sqrt{4^2 + 3^2} = 5$. The circle is $(x - 4)^2 + (y - 3)^2 = 25$. Setting $y = 0$ gives $(x - 4)^2 = 16$, so $x = 0$ or $x = 8$. The second intersection with the x-axis is at $x = 8$.

Thus, the correct answer is **D**.

7. The functions $\sin(x)$ and $\cos(x)$ are periodic with least period 2π. What is the least period of the function $\cos(\sin(x))$?

A $\frac{\pi}{2}$

B π

C 2π

D 4π

E It's not periodic.

Solution:

Since $\cos(\sin(x+\pi)) = \cos(-\sin(x)) = \cos(\sin(x))$, the function has period π. It cannot be smaller: $\cos(\sin(x)) = 1$ exactly when $\sin(x) = 0$, which happens only at integer multiples of π, so the maxima are spaced π apart. The least period is π.

Thus, the correct answer is **B**.

8. The ratio of the short side of a certain rectangle to the long side is equal to the ratio of the long side to the diagonal. What is the square of the ratio of the short side to the long side of this rectangle?

A $\frac{\sqrt{3}-1}{2}$

B $\frac{1}{2}$

C $\frac{\sqrt{5}-1}{2}$

D $\frac{\sqrt{2}}{2}$

E $\frac{\sqrt{6}-1}{2}$

Solution:

Let x and y be the short and long sides, so the diagonal is $\sqrt{x^2+y^2}$ and $\frac{x^2}{y^2} = \frac{y^2}{x^2+y^2}$. Writing $r = \frac{x^2}{y^2}$, the right side is $\frac{y^2}{x^2+y^2} = \frac{1}{r+1}$, so $r = \frac{1}{r+1}$, giving $r^2 + r - 1 = 0$. The positive root is $r = \frac{\sqrt{5}-1}{2}$.

Thus, the correct answer is **C**.

9. A circle has center $(-10, -4)$ and radius 13. Another circle has center $(3, 9)$ and radius $\sqrt{65}$. The line passing through the two points of intersection of the two circles has equation $x + y = c$. What is c?

A 3

B $3\sqrt{3}$

C $4\sqrt{2}$

D 6

E $\frac{13}{2}$

Solution:

The circles are $(x+10)^2 + (y+4)^2 = 169$ and $(x-3)^2 + (y-9)^2 = 65$. Expanding and subtracting the second from the first cancels the x^2 and y^2 terms and simplifies to $x + y = 3$. Any intersection point satisfies this, so it is the line through both, and $c = 3$.

Thus, the correct answer is **A**.

10. At Typico High School, 60% of the students like dancing, and the rest dislike it. Of those who like dancing, 80% say that they like it, and the rest say that they dislike it. Of those who dislike dancing, 90% say that they dislike it, and the rest say that they like it. What fraction of students who say they dislike dancing actually like it?

A 10%

B 12%

C 20%

D 25%

E $33\frac{1}{3}\%$

Solution:

Students who like dancing but say they dislike it make up $60\% \cdot 20\% = 12\%$ of all students. Students who dislike dancing and say so make up $40\% \cdot 90\% = 36\%$. Among everyone who says they dislike dancing, the fraction who actually like it is

$$\frac{12}{12+36} = \frac{12}{48} = \frac{1}{4} = 25\%.$$

Thus, the correct answer is **D**.

11. Call a positive integer *monotonous* if it is a one-digit number or its digits, when read from left to right, form either a strictly increasing or a strictly decreasing sequence. For example, $3, 23578,$ and 987620 are monotonous, but $88, 7434,$ and 23557 are not. How many monotonous positive integers are there?

- A 1024
- B 1524
- C 1533
- D 1536
- E 2048

Solution:

Strictly increasing monotonous numbers correspond to nonempty subsets of $\{1, \ldots, 9\}$, giving $2^9 - 1 = 511.$ Strictly decreasing ones correspond to subsets of $\{0, 1, \ldots, 9\}$ other than $\varnothing$ and $\{0\}$ (a leading 0 is not allowed), giving $2^{10} - 2 = 1022.$ The nine single-digit numbers are counted in both, so the total is $511 + 1022 - 9 = 1524.$

Thus, the correct answer is **B**.

12. What is the sum of the roots of $z^{12} = 64$ that have a positive real part?

A 2

B 4

C $2 + 2\sqrt{3}$

D $2\sqrt{2} + \sqrt{6}$

E $(1 + \sqrt{3}) + (1 + \sqrt{3})i$

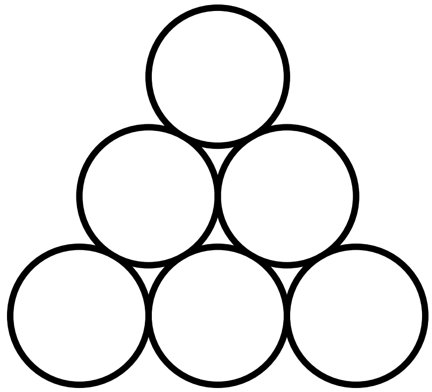
Solution:

The roots of $z^{12} = 64$ lie on the circle of radius $64^{1/12} = \sqrt{2}$, at angles that are multiples of 30°. Those with positive real part are at angles $0, \pm 30^\circ, \pm 60^\circ$. Their imaginary parts cancel, so the sum is

$$\sqrt{2} + 2\sqrt{2}\cos 30^\circ + 2\sqrt{2}\cos 60^\circ = \sqrt{2}\left(1 + \sqrt{3} + 1\right) = 2\sqrt{2} + \sqrt{6}.$$

Thus, the correct answer is **D**.

13. In the figure below, 3 of the 6 disks are to be painted blue, 2 are to be painted red, and 1 is to be painted green. Two paintings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same. How many different paintings are possible?



A 6

B 8

C 9

D 12

E 15

Solution:

The figure has 3 corner disks and 3 non-corner disks, with the symmetry group of a triangle. Fix the green disk's type. If green is a corner, the two red disks can be arranged so that both, one, or neither is adjacent to green, giving $1 + 3 + 2 = 6$ distinct paintings. If green is a non-corner, the two reds can have both, one, or neither in a corner, again $1 + 3 + 2 = 6$ paintings. The blue disks fill the rest, so the total is $6 + 6 = 12$.

Thus, the correct answer is **D**.

14. An ice-cream novelty item consists of a cup in the shape of a 4-inch-tall frustum of a right circular cone, with a 2-inch-diameter base at the bottom and a 4-inch-diameter base at the top, packed solid with ice cream, together with a solid cone of ice cream of height 4 inches, whose base, at the bottom, is the top base of the frustum. What is the total volume of the ice cream, in cubic inches?

A 8π

B $\frac{28\pi}{3}$

C 12π

D 14π

E $\frac{44\pi}{3}$

Solution:

Extending the frustum's sides to a point, similar triangles show the frustum equals a cone of radius 2 and height 8 minus a cone of radius 1 and height 4 :

$$\tfrac{1}{3}\pi(2^2)(8)-\tfrac{1}{3}\pi(1^2)(4)=\tfrac{32}{3}\pi-\tfrac{4}{3}\pi=\tfrac{28}{3}\pi.$$

The top cone of radius 2 and height 4 adds $\frac{1}{3}\pi(2^2)(4)=\frac{16}{3}\pi$. The total is $\frac{28}{3}\pi+\frac{16}{3}\pi=\frac{44}{3}\pi$.

Thus, the correct answer is **E**.

15. Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

- A $9 : 1$
- B $16 : 1$
- C $25 : 1$
- D $36 : 1$
- E $37 : 1$

Solution:

Let $X = [\triangle ABC]$, and draw segments CB', AC', and BA'. Triangle $BB'C$ has base $BB' = 3 \cdot AB$ and the same altitude as $\triangle ABC$ from C to line AB, so its area is $3X$; likewise $\triangle CC'A$ and $\triangle AA'B$ each have area $3X$. Next, $\triangle AA'C'$ has 3 times the base and the same height as $\triangle ACC'$, so its area is $9X$; similarly $\triangle CC'B'$ and $\triangle BB'A'$ each have area $9X$. Thus

$$[\triangle A'B'C'] = X + 3(3X) + 3(9X) = 37X,$$

so the ratio is $37 : 1$.

Thus, the correct answer is **E**.

16. The number $21! = 51{,}090{,}942{,}171{,}709{,}440{,}000$ has over $60{,}000$ positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

A $\frac{1}{21}$

B $\frac{1}{19}$

C $\frac{1}{18}$

D $\frac{1}{2}$

E $\frac{11}{21}$

Solution:

The exponent of 2 in $21!$ is $\lfloor 21/2 \rfloor + \lfloor 21/4 \rfloor + \lfloor 21/8 \rfloor + \lfloor 21/16 \rfloor = 10 + 5 + 2 + 1 = 18$. Every divisor has the form $2^i b$ with $0 \le i \le 18$ and b odd; it is odd exactly when $i = 0$. So the fraction of odd divisors is $\frac{1}{18+1} = \frac{1}{19}$.

Thus, the correct answer is **B**.

17. A coin is biased in such a way that on each toss the probability of heads is $\dfrac{2}{3}$ and the probability of tails is $\dfrac{1}{3}$. The outcomes of the tosses are independent. A player has the choice of playing Game A or Game B. In Game A she tosses the coin three times and wins if all three outcomes are the same. In Game B she tosses the coin four times and wins if both the outcomes of the first and second tosses are the same and the outcomes of the third and fourth tosses are the same. How do the chances of winning Game A compare to the chances of winning Game B?

A. The probability of winning Game A is $\dfrac{4}{81}$ less than the probability of winning Game B.

B. The probability of winning Game A is $\dfrac{2}{81}$ less than the probability of winning Game B.

C. The probabilities are the same.

D. The probability of winning Game A is $\dfrac{2}{81}$ greater than the probability of winning Game B.

E. The probability of winning Game A is $\dfrac{4}{81}$ greater than the probability of winning Game B.

Solution:

Let $p = \dfrac{2}{3}$. Game A is won when all three tosses match: $p^3 + (1-p)^3$. Game B needs the first pair to match and the second pair to match, each with probability $p^2 + (1-p)^2$, so the win probability is $\left(p^2 + (1-p)^2\right)^2$. With $p = \frac{2}{3}$, Game A gives $\left(\frac{2}{3}\right)^3 + \left(\frac{1}{3}\right)^3 = \frac{9}{27} = \frac{1}{3}$, and Game B gives $\left(\frac{4}{9} + \frac{1}{9}\right)^2 = \left(\frac{5}{9}\right)^2 = \frac{25}{81}$. The difference is $\frac{27}{81} - \frac{25}{81} = \frac{2}{81}$, so Game A is $\frac{2}{81}$ more likely.

Thus, the correct answer is **D**.

18. The diameter $\overline{AB}$ of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment $\overline{AE}$ intersects the circle at a point C between A and E. What is the area of $\triangle ABC$?

A $\dfrac{120}{37}$

B $\dfrac{140}{39}$

C $\dfrac{145}{39}$

D $\dfrac{140}{37}$

E $\dfrac{120}{31}$

Solution:

Since $\angle ACB$ is inscribed in a semicircle, it is a right angle, so $\triangle ABC \sim \triangle AED$ (both right-angled and sharing angle A). Their areas are in ratio $AB^2 : AE^2$. Here $AB = 4$, so $AB^2 = 16$, and $AD = AB + BD = 7$, so $AE^2 = AD^2 + ED^2 = 49 + 25 = 74$. The area of $\triangle AED$ is $\frac{1}{2} \cdot 7 \cdot 5 = \frac{35}{2}$. Thus

$$[\triangle ABC] = \frac{16}{74} \cdot \frac{35}{2} = \frac{140}{37}.$$

Thus, the correct answer is **D**.

19. Let $N = 123456789101112\dots4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

A 1

B 4

C 9

D 18

E 44

Solution:

The last digit of N is $4,$ so $N \equiv 4 \pmod 5$. For mod $9,$ sum the digits: the numbers 1–9 contribute their digits, the tens digits of 10–44 and the units digits together sum to $270,$ which is a multiple of $9,$ so $N \equiv 0 \pmod 9$. The number $N - 9$ is then a multiple of $9,$ and its last digit is $5,$ so it is a multiple of $5;$ hence $N - 9$ is a multiple of $45.$ Therefore $N \equiv 9 \pmod{45}$.

Thus, the correct answer is **C**.

20. Real numbers x and y are chosen independently and uniformly at random from the interval $(0, 1)$. What is the probability that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$, where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to the real number r?

A $\frac{1}{8}$

B $\frac{1}{6}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

For each positive integer n, $\lfloor \log_2 x \rfloor = -n$ exactly when $\frac{1}{2^n} \leq x < \frac{1}{2^{n-1}}$, an interval of length $\frac{1}{2^n}$. The event that both floors equal $-n$ is a square of area $\frac{1}{4^n}$. Summing over all n, the probability is

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1/4}{1 - 1/4} = \frac{1}{3}.$$

Thus, the correct answer is **D**.

21. Last year Isabella took 7 math tests and received 7 different scores, each an integer between 91 and $100,$ inclusive. After each test she noticed that the average of her test scores was an integer. Her score on the seventh test was $95.$ What was her score on the sixth test?

A 92

B 94

C 96

D 98

E 100

Solution:

Let S be the sum of all seven scores. Then S is a multiple of 7 with $658 \leq S \leq 679,$ so $S \in \{658, 665, 672, 679\}.$ Since the average after six tests is an integer, $S - 95$ is a multiple of $6,$ which forces $S = 665.$ Then the first six scores sum to $570,$ a multiple of $5;$ the average after five tests is an integer, so the first five scores also sum to a multiple of $5,$ making the sixth score a multiple of $5.$ Since all scores differ and the seventh is $95,$ the sixth must be $100.$

Thus, the correct answer is **E**.

22. Abby, Bernardo, Carl, and Debra play a game in which each of them starts with four coins. The game consists of four rounds. In each round, four balls are placed in an urn—one green, one red, and two white. The players each draw a ball at random without replacement. Whoever gets the green ball gives one coin to whoever gets the red ball. What is the probability that, at the end of the fourth round, each of the players has four coins?

A $\frac{7}{576}$

B $\frac{5}{192}$

C $\frac{1}{36}$

D $\frac{5}{144}$

E $\frac{7}{48}$

Solution:

Each round has $4 \cdot 3 = 12$ equally likely (giver, receiver) pairs, so there are 12^4 outcome sequences. Everyone ends with four coins exactly when the four transfers cancel. The favorable patterns are: a 4-cycle of gifts ($24 \cdot 6 = 144$ ways), two disjoint mutual exchanges ($24 \cdot 3 = 72$), one pair exchanging twice ($6 \cdot 6 = 36$), and one player both giving to and receiving from each of two others ($4 \cdot 3 \cdot 24 = 288$). These total $144 + 72 + 36 + 288 = 540$. The probability is $\frac{540}{12^4} = \frac{540}{20736} = \frac{5}{192}$.

Thus, the correct answer is **B**.

23. The graph of $y = f(x)$, where $f(x)$ is a polynomial of degree 3, contains points $A(2,4), B(3,9)$, and $C(4,16)$. Lines AB, AC, and BC intersect the graph again at points D, E, and F, respectively, and the sum of the x-coordinates of D, E, and F is 24. What is $f(0)$?

A -2

B 0

C 2

D $\frac{24}{5}$

E 8

Solution:

The points A, B, C lie on $y = x^2$, so $g(x) = f(x) - x^2$ has roots $2, 3, 4 : g(x) = a(x-2)(x-3)(x-4)$ for some $a \neq 0$. The coefficients of x^3 and x^2 in f are a and $1 - 9a$, so by Vieta the three roots of $f(x) - L(x)$ (for any linear L) sum to $9 - \frac{1}{a}$. The lines AB, AC, BC meet the cubic in triples $\{2, 3, x_D\}, \{2, 4, x_E\}, \{3, 4, x_F\}$, so

$$x_D + x_E + x_F = 3\left(9 - \tfrac{1}{a}\right) - 2(2+3+4) = 9 - \tfrac{3}{a} = 24,$$

giving $a = -\frac{1}{5}$. Then $f(x) = x^2 - \frac{1}{5}(x-2)(x-3)(x-4)$, so $f(0) = 0 - \frac{1}{5}(-2)(-3)(-4) = \frac{24}{5}$.

Thus, the correct answer is **D**.

24. Quadrilateral $ABCD$ has right angles at B and C, $\triangle ABC \sim \triangle BCD$, and $AB > BC$. There is a point E in the interior of $ABCD$ such that $\triangle ABC \sim \triangle CEB$ and the area of $\triangle AED$ is 17 times the area of $\triangle CEB$. What is $\dfrac{AB}{BC}$?

- A $1+\sqrt{2}$
- B $2+\sqrt{2}$
- C $\sqrt{17}$
- D $2+\sqrt{5}$
- E $1+2\sqrt{3}$

Solution:

Set $BC = 1$ and $AB = r > 1$. The similarity $\triangle ABC \sim \triangle BCD$ with the right angles places the figure at $C = (0,0), B = (0,1), A = (r,1), D = \left(\frac{1}{r}, 0\right)$. Let $E = (x,y)$ with $x, y > 0$. From $\triangle ABC \sim \triangle CEB$ we get $\dfrac{x}{y} = \tan(\angle ECB) = \tan(\angle BAC) = \dfrac{1}{r}$ and $x^2 + y^2 = \dfrac{r}{1+r^2}$, so $x = \dfrac{r}{1+r^2}, y = \dfrac{r^2}{1+r^2}$. The area of $\triangle CEB$ is $\frac{1}{2}x$, and computing $[\triangle AED]$ by the shoelace formula and setting $[\triangle AED] = 17[\triangle CEB]$ simplifies to $r^4 - 18r^2 + 1 = 0$. Then $r^2 = 9 + 4\sqrt{5} = (2+\sqrt{5})^2$, so $r = 2 + \sqrt{5}$.

Thus, the correct answer is **D**.

25. A set of n people participate in an online video basketball tournament. Each person may be a member of any number of 5-player teams, but no two teams may have exactly the same 5 members. The site statistics show a curious fact: The average, over all subsets of size 9 of the set of n participants, of the number of complete teams whose members are among those 9 people is equal to the reciprocal of the average, over all subsets of size 8 of the set of n participants, of the number of complete teams whose members are among those 8 people. How many values $n, 9 \leq n \leq 2017,$ can be the number of participants?

A 477

B 482

C 487

D 557

E 562

Solution:

Let T be the number of teams. Summing over size-9 subsets counts each team $\binom{n-5}{4}$ times and over size-8 subsets $\binom{n-5}{3}$ times. The averages are $\frac{\binom{n-5}{4}T}{\binom{n}{9}}$ and $\frac{\binom{n-5}{3}T}{\binom{n}{8}}$; setting the first equal to the reciprocal of the second and simplifying gives

$$T = \frac{n(n-1)(n-2)(n-3)(n-4)}{2^5 \cdot 3^2 \cdot 5 \cdot 7}.$$

We need this to be a positive integer with $n \geq 9$. Let $N = n(n-1)(n-2)(n-3)(n-4)$; as a product of five consecutive integers, N is always divisible by 5. Checking residues, $7 \mid N, 9 \mid N,$ and $32 \mid N$ each hold for a fixed set of residues, giving $5 \cdot 7 \cdot 8 = 280$ solutions modulo 1008. So there are 560 values in $1 \leq n \leq 2016$; removing $n = 1, 2, 3, 4$ (which are below 9) and adding $n = 2017$ (since $2017 \equiv 1 \pmod{1008}$) gives $560 - 4 + 1 = 557$ valid values.

Thus, the correct answer is **D**.

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