2017 AMC 12A Solutions

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1. Pablo buys popsicles for his friends. The store sells single popsicles for $\$1$ each, 3-popsicle boxes for $\$2$, and 5-popsicle boxes for $\$3$. What is the greatest number of popsicles that Pablo can buy with $\$8$?

A 8

B 11

C 12

D 13

E 15

Solution:

The cheapest popsicles come from the 5-popsicle box, at $\frac{\$3}{5} = \$0.60$ each. Even at that rate, 14 popsicles would cost $14 \cdot \$0.60 = \$8.40$, more than $\$8$.

So Pablo can buy at most 13, and he achieves this with two 5-boxes for $\$6$ and one 3-box for $\$2$, giving $2 \cdot 5 + 3 = 13$ popsicles.

Thus, the correct answer is **D**.

2. The sum of two nonzero real numbers is 4 times their product. What is the sum of the reciprocals of the two numbers?

A 1

B 2

C 4

D 8

E 12

Solution:

Let the numbers be x and y, so $x + y = 4xy$.

Dividing both sides by xy gives

$$\frac{x+y}{xy} = 4,$$

and the left side is exactly $\frac{1}{y} + \frac{1}{x}$. So the sum of the reciprocals is 4.

Thus, the correct answer is **C**.

3. Ms. Carroll promised that anyone who got all the multiple choice questions right on the upcoming exam would receive an A on the exam. Which one of these statements necessarily follows logically?

A. If Lewis did not receive an A, then he got all of the multiple choice questions wrong.

B. If Lewis did not receive an A, then he got at least one of the multiple choice questions wrong.

C. If Lewis got at least one of the multiple choice questions wrong, then he did not receive an A.

D. If Lewis received an A, then he got all of the multiple choice questions right.

E. If Lewis received an A, then he got at least one of the multiple choice questions right.

Solution:

The promise is "all right $\Rightarrow$ A." An implication is equivalent only to its contrapositive: "not A $\Rightarrow$ not all right."

"Not all right" means at least one question was wrong, which is exactly statement B. The converse and inverse do not follow, and getting "all wrong" is a much stronger claim than the negation.

Thus, the correct answer is **B**.

4. Jerry and Silvia wanted to go from the southwest corner of a square field to the northeast corner. Jerry walked due east and then due north to reach the goal, but Silvia headed northeast and reached the goal walking in a straight line. Which of the following is closest to how much shorter Silvia's trip was, compared to Jerry's trip?

- A 30%
- B 40%
- C 50%
- D 60%
- E 70%

Solution:

If the square has side x, Jerry walks $x+x=2x$, while Silvia walks the diagonal $\sqrt{x^2+x^2}=x\sqrt{2}$.

The fraction by which Silvia's trip is shorter is

$$\frac{2x-x\sqrt{2}}{2x}=1-\frac{\sqrt{2}}{2}\approx 1-0.707=0.293.$$

This is closest to 30%.

Thus, the correct answer is **A**.

5. At a gathering of 30 people, there are 20 people who all know each other and 10 people who know no one. People who know each other hug, and people who do not know each other shake hands. How many handshakes occur?

- A 240
- B 245
- C 290
- D 480
- E 490

Solution:

Each of the 20 people who know each other shakes hands with only the 10 strangers. Each of the 10 strangers shakes hands with all 29 other people.

Summing handshake counts and dividing by 2 (each handshake involves two people) gives

$$\frac{1}{2}(20 \cdot 10 + 10 \cdot 29) = \frac{1}{2}(200 + 290) = 245.$$

Thus, the correct answer is **B**.

6. Joy has 30 thin rods, one each of every integer length from 1 cm through 30 cm. She places the rods with lengths 3 cm, 7 cm, and 15 cm on a table. She then wants to choose a fourth rod that she can put with these three to form a quadrilateral with positive area. How many of the remaining rods can she choose as the fourth rod?

- A 16
- B 17
- C 18
- D 19
- E 20

Solution:

Four lengths form a quadrilateral with positive area if and only if the longest is strictly less than the sum of the other three. With a fourth rod of length n, this requires $15 < 3 + 7 + n$ and $n < 3 + 7 + 15$, so

$$5 < n < 25.$$

The integers from 6 to 24 give 19 values, but the rods of length 7 and 15 are already on the table, leaving $19 - 2 = 17$ choices.

Thus, the correct answer is **B**.

7. Define a function on the positive integers recursively by $f(1) = 2, f(n) = f(n-1) + 1$ if n is even, and $f(n) = f(n-2) + 2$ if n is odd and greater than 1. What is $f(2017)$?

A 2017

B 2018

C 4034

D 4035

E 4036

Solution:

Listing values: $f(1) = 2, f(2) = f(1) + 1 = 3, f(3) = f(1) + 2 = 4, f(4) = f(3) + 1 = 5,$ suggesting $f(n) = n + 1.$

Both rules are consistent with $f(n) = n + 1:$ for even $n, (n-1) + 1 + 1 = n + 1,$ and for odd $n, (n-2) + 1 + 2 = n + 1.$ Since the recursion determines f uniquely, $f(2017) = 2018.$

Thus, the correct answer is **B**.

8. The region consisting of all points in three-dimensional space within 3 units of line segment AB has volume 216π. What is the length AB?

- A 6
- B 12
- C 18
- D 20
- E 24

Solution:

Let $h = AB$. The region is a cylinder of radius 3 and height h with a hemisphere of radius 3 on each end.

The cylinder has volume $\pi \cdot 3^2 \cdot h = 9\pi h$, and the two hemispheres together form a sphere of volume $\frac{4}{3}\pi \cdot 3^3 = 36\pi$. So

$$9\pi h + 36\pi = 216\pi,$$

giving $h = 20$.

Thus, the correct answer is **D**.

9. Let S be the set of points (x, y) in the coordinate plane such that two of the three quantities $3, x+2,$ and $y-4$ are equal and the third of the three quantities is no greater than this common value. Which of the following is a correct description of S?

A a single point

B two intersecting lines

C three lines whose pairwise intersections are three distinct points

D a triangle

E three rays with a common endpoint

Solution:

Consider which two of $3, x+2, y-4$ are the (equal) larger pair.

If $3 = x+2 \geq y-4$: then $x = 1$ and $y \leq 7$, a downward ray from $(1, 7)$. If $3 = y-4 \geq x+2$: then $y = 7$ and $x \leq 1$, a leftward ray from $(1, 7)$. If $x+2 = y-4 \geq 3$: then $y = x+6$ and $x \geq 1$, a ray from $(1, 7)$ going up and to the right.

All three rays share the endpoint $(1, 7)$, so S is three rays with a common endpoint.

Thus, the correct answer is **E**.

10. Chloé chooses a real number uniformly at random from the interval $[0, 2017]$. Independently, Laurent chooses a real number uniformly at random from the interval $[0, 4034]$. What is the probability that Laurent's number is greater than Chloé's number?

A $\frac{1}{2}$

B $\frac{2}{3}$

C $\frac{3}{4}$

D $\frac{5}{6}$

E $\frac{7}{8}$

Solution:

With probability $\frac{1}{2}$, Laurent's number lies in $[2017, 4034]$, which exceeds any number Chloé could choose, so he wins for certain.

With the other probability $\frac{1}{2}$, Laurent's number lies in $[0, 2017]$, matching Chloé's interval; by symmetry he is larger half the time. The total probability is

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

Thus, the correct answer is **C**.

11. Claire adds the degree measures of the interior angles of a convex polygon and arrives at a sum of 2017. She then discovers that she forgot to include one angle. What is the degree measure of the forgotten angle?

A 37

B 63

C 117

D 143

E 163

Solution:

If the polygon has n sides and the forgotten angle is α, then $(n-2)180 = 2017 + \alpha$. Since $0 < \alpha < 180$,

$$2017 < (n-2)180 < 2197.$$

The only multiple of 180 in this range is $2160 = (14-2)180$, so $n = 14$ and

$$\alpha = 2160 - 2017 = 143.$$

Thus, the correct answer is **D**.

12. There are 10 horses, named Horse 1, Horse $2, \ldots,$ Horse 10. They get their names from how many minutes it takes them to run one lap around a circular race track: Horse k runs one lap in exactly k minutes. At time 0 all the horses are together at the starting point on the track. The horses start running in the same direction, and they keep running around the circular track at their constant speeds. The least time $S > 0$, in minutes, at which all 10 horses will again simultaneously be at the starting point is $S = 2520$. Let $T > 0$ be the least time, in minutes, such that at least 5 of the horses are again at the starting point. What is the sum of the digits of T?

- A 2
- B 3
- C 4
- D 5
- E 6

Solution:

Horse k is at the starting point at time t precisely when $k \mid t$. So we want the smallest t with at least 5 divisors among $1, 2, \ldots, 10$.

Checking small values, $t = 12$ is divisible by $1, 2, 3, 4,$ and 6, giving exactly 5 such horses, and no smaller t reaches 5. Thus $T = 12$, and the sum of its digits is $1 + 2 = 3$.

Thus, the correct answer is **B**.

13. Driving at a constant speed, Sharon usually takes 180 minutes to drive from her house to her mother's house. One day Sharon begins the drive at her usual speed, but after driving $\frac{1}{3}$ of the way, she hits a bad snowstorm and reduces her speed by 20 miles per hour. This time the trip takes her a total of 276 minutes. How many miles is the drive from Sharon's house to her mother's house?

- A 132
- B 135
- C 138
- D 141
- E 144

Solution:

Let the distance be d miles and the usual speed r mph. Since the usual trip is 3 hours, $d = 3r$.

The first $\frac{1}{3}$ of the drive takes $\frac{1}{3} \cdot 180 = 60$ minutes at speed r, so the remaining $\frac{2}{3}$ takes $276 - 60 = 216$ minutes $= \frac{18}{5}$ hours at speed $r - 20$.

That final portion covers $\frac{2}{3}d = 2r$ miles, so

$$2r = (r - 20) \cdot \frac{18}{5}.$$

Solving gives $10r = 18r - 360$, so $r = 45$ and $d = 3 \cdot 45 = 135$.

Thus, the correct answer is **B**.

14. Alice refuses to sit next to either Bob or Carla. Derek refuses to sit next to Eric. How many ways are there for the five of them to sit in a row of 5 chairs under these conditions?

A 12

B 16

C 28

D 32

E 40

Solution:

Let X, Y, Z be the seatings where Alice-Bob, Alice-Carla, and Derek-Eric are adjacent, respectively. The answer is $5! - |X \cup Y \cup Z|$.

Treating a forbidden pair as a block gives $|X| = |Y| = |Z| = 2 \cdot 4! = 48$. For intersections, $|X \cap Y| = 2 \cdot 3! = 12$ (Alice between Bob and Carla), $|X \cap Z| = |Y \cap Z| = 2 \cdot 2 \cdot 3! = 24$, and $|X \cap Y \cap Z| = 2 \cdot 2 \cdot 2! = 8$.

By inclusion-exclusion, $|X \cup Y \cup Z| = (48 \cdot 3) - (12 + 24 + 24) + 8 = 92$, so the answer is $120 - 92 = 28$.

Thus, the correct answer is **C**.

15. Let $f(x) = \sin x + 2\cos x + 3\tan x$, using radian measure for the variable x. In what interval does the smallest positive value of x for which $f(x) = 0$ lie?

A $(0, 1)$

B $(1, 2)$

C $(2, 3)$

D $(3, 4)$

E $(4, 5)$

Solution:

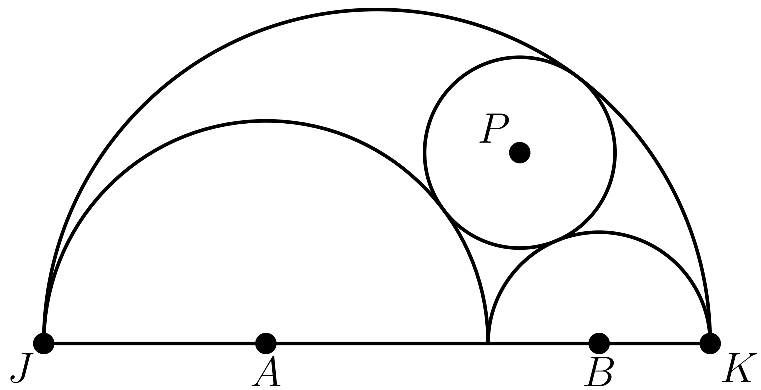
For $0 < x < \frac{\pi}{2}$ all three terms are positive, so $f(x) > 0$. For $\frac{\pi}{2} < x < \pi$, $\tan x$ is negative and dominates, keeping $f(x) < 0$. So no root occurs before $x = \pi$.

At $x = \pi$, $f(\pi) = 0 + 2(-1) + 0 = -2 < 0$. At $x = \frac{5\pi}{4}$, $\tan x = 1$ so $f = -\frac{\sqrt{2}}{2} + 3 > 0$. By the intermediate value theorem the smallest positive root lies in $\left(\pi, \frac{5\pi}{4}\right)$.

Since $\pi > 3$ and $\frac{5\pi}{4} < 4$, this interval sits inside $(3, 4)$.

Thus, the correct answer is **D**.

16. In the figure below, semicircles with centers at A and B and with radii 2 and 1, respectively, are drawn in the interior of, and sharing bases with, a semicircle with diameter $\overline{JK}$. The two smaller semicircles are externally tangent to each other and internally tangent to the largest semicircle. A circle centered at P is drawn externally tangent to the two smaller semicircles and internally tangent to the largest semicircle. What is the radius of the circle centered at P?



A $\frac{3}{4}$

B $\frac{6}{7}$

C $\frac{1}{2}\sqrt{3}$

D $\frac{5}{8}\sqrt{2}$

E $\frac{11}{12}$

Solution:

The large semicircle has radius 3 and center C, the midpoint of $\overline{JK}$. Placing J at the origin, $A = 2, B = 5, C = 3, K = 6$ along the base. Let r be the radius of the circle at P.

By tangency, $PA = 2 + r$, $PB = 1 + r$, and $PC = 3 - r$. Dropping a perpendicular from P to the base at horizontal position $3 + x$ with height h, the Pythagorean theorem gives

$$h^2 = (2+r)^2 - (1+x)^2 = (3-r)^2 - x^2 = (1+r)^2 - (2-x)^2.$$

These reduce to two linear equations in r and x, whose solution is $r = \frac{6}{7}$ (and $x = \frac{9}{7}$).

Thus, the correct answer is **B**.

17. There are 24 different complex numbers z such that $z^{24} = 1$. For how many of these is z^6 a real number?

- A 0
- B 4
- C 6
- D 12
- E 24

Solution:

The 24 solutions are the 24th roots of unity, $z = e^{\pi ik/12}$ for $k = 0, 1, \ldots, 23$.

Then $z^6 = e^{\pi ik/2} = \cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2}$, which is real exactly when $\sin\frac{k\pi}{2} = 0$, i.e. when k is even. There are 12 even values of k in the range.

Thus, the correct answer is **D**.

18. Let $S(n)$ equal the sum of the digits of positive integer n. For example, $S(1507) = 13$. For a particular positive integer n, $S(n) = 1274$. Which of the following could be the value of $S(n+1)$?

A 1

B 3

C 12

D 1239

E 1265

Solution:

Adding 1 to n increases the digit sum by 1, except that each trailing 9 turns into a 0, losing 9. If n ends in exactly k nines, then $S(n+1) = S(n) + 1 - 9k = 1275 - 9k$.

So the possible values are $1275, 1266, 1257, \ldots$ Among the choices, only $1239 = 1275 - 9 \cdot 4$ fits (for example, n ending in four 9s preceded by enough 1s).

Thus, the correct answer is **D**.

19. A square with side length x is inscribed in a right triangle with sides of length $3, 4,$ and 5 so that one vertex of the square coincides with the right-angle vertex of the triangle. A square with side length y is inscribed in another right triangle with sides of length $3, 4,$ and 5 so that one side of the square lies on the hypotenuse of the triangle. What is $\frac{x}{y}$?

A $\frac{12}{13}$

B $\frac{35}{37}$

C 1

D $\frac{37}{35}$

E $\frac{13}{12}$

Solution:

For the first square, the two smaller triangles it cuts off are similar to the whole triangle, giving $\frac{x}{3-x} = \frac{4-x}{x}$, so $x = \frac{12}{7}$. (Equivalently, a square in the right angle has side $\frac{3 \cdot 4}{3+4}$.)

For the second square, take the hypotenuse of length 5 as base; the altitude to it is $h = \frac{3 \cdot 4}{5} = \frac{12}{5}$. A square with a side on a base b and height h has side $\frac{bh}{b+h}$, so

$$y = \frac{5 \cdot \frac{12}{5}}{5 + \frac{12}{5}} = \frac{12}{\frac{37}{5}} = \frac{60}{37}.$$

Therefore $\frac{x}{y} = \frac{12}{7} \cdot \frac{37}{60} = \frac{37}{35}$.

Thus, the correct answer is **D**.

20. How many ordered pairs (a, b) such that a is a positive real number and b is an integer between 2 and 200, inclusive, satisfy the equation $(\log_b a)^{2017} = \log_b(a^{2017})$?

- A 198
- B 199
- C 398
- D 399
- E 597

Solution:

Let $u = \log_b a$. Since $\log_b(a^{2017}) = 2017 \log_b a$, the equation is $u^{2017} = 2017u$, so $u = 0$ or $u^{2016} = 2017$.

If $u = 0$, then $a = 1$, valid for every one of the 199 bases. If $u^{2016} = 2017$, then $u = \pm 2017^{1/2016}$, giving 2 values of a for each base, i.e. $2 \cdot 199 = 398$ pairs.

In total there are $199 + 398 = 597$ ordered pairs.

Thus, the correct answer is **E**.

21. A set S is constructed as follows. To begin, $S = \{0, 10\}$. Repeatedly, as long as possible, if x is an integer root of some polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ for some $n \geq 1$, all of whose coefficients a_i are elements of S, then x is put into S. When no more elements can be added to S, how many elements does S have?

- A 4
- B 5
- C 7
- D 9
- E 11

Solution:

Using $10x + 10$, the root -1 enters S. Then 1 enters as a root of $-x^{10} - x^9 - \cdots - x + 10$, and -10 enters from $x + 10$.

Now $x^3 + x - 10$ has root 2, and $x + 2$ gives -2; then $2x - 10$ and $2x + 10$ give ± 5. At this point $S = \{0, \pm 1, \pm 2, \pm 5, \pm 10\}$.

No further integer can appear: by the Rational Root Theorem any integer root divides the constant term, which is always a factor of 10. So S has 9 elements.

Thus, the correct answer is **D**.

22. A square is drawn in the Cartesian coordinate plane with vertices at $(2, 2)$, $(-2, 2)$, $(-2, -2)$, and $(2, -2)$. A particle starts at $(0, 0)$. Every second it moves with equal probability to one of the eight lattice points closest to its current position, independently of its previous moves. In other words, the probability is $\frac{1}{8}$ that the particle will move from (x, y) to each of $(x, y+1)$, $(x+1, y+1)$, $(x+1, y)$, $(x+1, y-1)$, $(x, y-1)$, $(x-1, y-1)$, $(x-1, y)$, or $(x-1, y+1)$. The particle will eventually hit the square for the first time, either at one of the 4 corners of the square or at one of the 12 lattice points in the interior of one of the sides of the square. The probability that it will hit at a corner rather than at an interior point of a side is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

A 4

B 5

C 7

D 15

E 39

Solution:

By symmetry, group the relevant interior points into three types: $C = \{(0,0)\}$, the "axis" points $A = \{(\pm1, 0), (0, \pm1)\}$, and the "diagonal" points $I = \{(\pm1, \pm1)\}$. Let a, c, i be the probabilities of eventually hitting a corner starting from a point of type A, C, I.

Reading off the transition probabilities (a point in A goes to A with prob $\frac{2}{8}$, to C with $\frac{1}{8}$, to I with $\frac{2}{8}$, and to a side interior with $\frac{3}{8}$, etc.) gives

$$a = \tfrac{2}{8}a + \tfrac{1}{8}c + \tfrac{2}{8}i, \quad c = \tfrac{4}{8}a + \tfrac{4}{8}i, \quad i = \tfrac{2}{8}a + \tfrac{1}{8}c + \tfrac{1}{8}.$$

Solving yields $a = \frac{1}{14}, c = \frac{4}{35}, i = \frac{11}{70}$. The required probability is $c = \frac{4}{35}$, so $m + n = 4 + 35 = 39$.

Thus, the correct answer is **E**.

23. For certain real numbers $a, b,$ and $c,$ the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of $g(x)$ is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is $f(1)$?

A -9009

B -8008

C -7007

D -6006

E -5005

Solution:

Since g has three distinct roots all shared by the quartic f, we can write $f(x) = (x - q)g(x)$ for some remaining root q. Expanding,

$$f(x) = x^4 + (a - q)x^3 + (1 - qa)x^2 + (10 - q)x - 10q.$$

Matching the x coefficient, $10 - q = 100$, so $q = -90$. Matching the x^3 coefficient, $a - q = 1$, so $a = -89$.

Then $g(1) = 1 + a + 1 + 10 = 12 - 89 = -77$ and $1 - q = 91$, so

$$f(1) = (1 - q)g(1) = 91 \cdot (-77) = -7007.$$

Thus, the correct answer is **C**.

24. Quadrilateral $ABCD$ is inscribed in circle O and has sides $AB = 3, BC = 2, CD = 6$, and $DA = 8$. Let X and Y be points on BD such that $\frac{DX}{BD} = \frac{1}{4}$ and $\frac{BY}{BD} = \frac{11}{36}$. Let E be the intersection of line AX and the line through Y parallel to AD. Let F be the intersection of line CX and the line through E parallel to AC. Let G be the point on circle O other than C that lies on line CX. What is $XF \cdot XG$?

A 17

B $\frac{59 - 5\sqrt{2}}{3}$

C $\frac{91 - 12\sqrt{3}}{4}$

D $\frac{67 - 10\sqrt{2}}{3}$

E 18

Solution:

Because $YE \parallel AD$ and $EF \parallel AC$, we get $\triangle XEY \sim \triangle XAD$ and $\triangle XEF \sim \triangle XAC$, giving $\frac{XY}{XE} = \frac{XD}{XA}$ and $\frac{XF}{XE} = \frac{XC}{XA}$. Hence $\frac{XC}{XD} = \frac{XF}{XY}$, so $XF \cdot XD = XC \cdot XY$.

Power of a Point at X gives $XC \cdot XG = XD \cdot XB$, and combining yields $XF \cdot XG = XB \cdot XY$. With $d = BD, DX = \frac{1}{4}d$ and $BY = \frac{11}{36}d$, so

$$XF \cdot XG = \left(d - \tfrac{1}{4}d\right)\left(d - \tfrac{1}{4}d - \tfrac{11}{36}d\right) = \frac{3}{4}d \cdot \frac{4}{9}d = \frac{d^2}{3}.$$

Since $ABCD$ is cyclic, $\angle BAD$ and $\angle BCD$ are supplementary. The Law of Cosines on $\triangle ABD$ and $\triangle CBD$ gives $\frac{73 - d^2}{48} = \frac{d^2 - 40}{24}$, so $d^2 = 51$. Therefore $XF \cdot XG = \frac{51}{3} = 17$.

Thus, the correct answer is **A**.

25. The vertices V of a centrally symmetric hexagon in the complex plane are given by

$$V = \left\{\sqrt{2}i,\ -\sqrt{2}i,\ \tfrac{1}{\sqrt{8}}(1+i),\ \tfrac{1}{\sqrt{8}}(-1+i),\ \tfrac{1}{\sqrt{8}}(1-i),\ \tfrac{1}{\sqrt{8}}(-1-i)\right\}.$$

For each $j, 1 \leq j \leq 12$, an element z_j is chosen from V at random, independently of the other choices. Let $P = \prod_{j=1}^{12} z_j$ be the product of the 12 numbers selected. What is the probability that $P = -1$?

A $\dfrac{5 \cdot 11}{3^{10}}$

B $\dfrac{5^2 \cdot 11}{2 \cdot 3^{10}}$

C $\dfrac{5 \cdot 11}{3^9}$

D $\dfrac{5 \cdot 7 \cdot 11}{2 \cdot 3^{10}}$

E $\dfrac{2^2 \cdot 5 \cdot 11}{3^{10}}$

Solution:

Let $A = \{\sqrt{2}i, -\sqrt{2}i\}$ (each of magnitude $\sqrt{2}$) and B be the other four elements (each of magnitude $\dfrac{1}{2}$). Since $|P| = (\sqrt{2})^{\#A}\left(\frac{1}{2}\right)^{\#B} = 1$ forces $\#A = 8$ and $\#B = 4$, exactly 8 factors must come from A and 4 from B.

A product of 8 elements of A equals ± 16 (real), and a product of 4 elements of B equals one of $\pm\frac{1}{16}, \pm\frac{i}{16}$. Their product is one of $\pm 1, \pm i$, each equally likely, so exactly $\frac{1}{4}$ of these configurations give $P = -1$.

The chance of landing in the 8-from-A, 4-from-B pattern is $\binom{12}{4}\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^4 = \dfrac{880}{3^{10}}$. Multiplying by $\frac{1}{4}$ gives

$$P = \frac{1}{4} \cdot \frac{880}{3^{10}} = \frac{220}{3^{10}} = \frac{2^2 \cdot 5 \cdot 11}{3^{10}}.$$

Thus, the correct answer is **E**.

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