2016 AMC 12B Solutions

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1. What is the value of

$$\frac{2a^{-1}+\frac{a^{-1}}{2}}{a}$$

when $a=\frac{1}{2}$?

A 1

B 2

C $\frac{5}{2}$

D 10

E 20

Solution:

With $a=\frac{1}{2}$, we have $a^{-1}=2$. The numerator is $2\cdot 2+\dfrac{2}{2}=4+1=5$, and dividing by $a=\frac{1}{2}$ gives $\dfrac{5}{1/2}=10$.

Thus, the correct answer is **D**.

2. The harmonic mean of two numbers can be computed as twice their product divided by their sum. The harmonic mean of 1 and 2016 is closest to which integer?

- A 2
- B 45
- C 504
- D 1008
- E 2015

Solution:

The harmonic mean is $\frac{2 \cdot 1 \cdot 2016}{1 + 2016} = \frac{4032}{2017}$. Since $\frac{2016}{2017}$ is very close to 1, this is just under 2, so the closest integer is 2.

Thus, the correct answer is **A**.

3. Let $x = -2016$. What is the value of

$$\Bigg\vert\Big\vert |x|-x\Big\vert-|x|\Bigg\vert-x?$$

- A -2016
- B 0
- C 2016
- D 4032
- E 6048

Solution:

Since $x = -2016$, $|x| = 2016$. The innermost expression is $|x| - x = 2016 + 2016 = 4032$. Then $\big|4032\big| - |x| = 4032 - 2016 = 2016$, and the outer absolute value leaves 2016. Finally subtracting x gives $2016 - (-2016) = 4032$.

Thus, the correct answer is **D**.

4. The ratio of the measures of two acute angles is $5 : 4$, and the complement of one of these two angles is twice as large as the complement of the other. What is the sum of the degree measures of the two angles?

A 75

B 90

C 135

D 150

E 270

Solution:

Let the angles be $\alpha < \beta$ with $\beta = \frac{5}{4}\alpha$. The larger complement belongs to the smaller angle, so $90 - \alpha = 2(90 - \beta) = 180 - \frac{5}{2}\alpha$. This gives $\frac{3}{2}\alpha = 90$, so $\alpha = 60^\circ$ and $\beta = 75^\circ$. The sum is 135°.

Thus, the correct answer is **C**.

5. The War of 1812 started with a declaration of war on Thursday, June 18, 1812. The peace treaty to end the war was signed 919 days later, on December 24, 1814. On what day of the week was the treaty signed?

- A Friday
- B Saturday
- C Sunday
- D Monday
- E Tuesday

Solution:

Because $919 = 7 \cdot 131 + 2$, the treaty was signed 131 full weeks plus 2 days after Thursday. Two days beyond Thursday is Saturday.

Thus, the correct answer is **B**.

6. All three vertices of $\triangle ABC$ lie on the parabola defined by $y = x^2$, with A at the origin and $\overline{BC}$ parallel to the x-axis. The area of the triangle is 64. What is the length of BC?

A	4
B	6
C	8
D	10
E	16

Solution:

Let the vertex in the first quadrant be (x, x^2). By symmetry the base is $BC = 2x$ and the height is x^2, so $\frac{1}{2} \cdot 2x \cdot x^2 = x^3 = 64$. Thus $x = 4$ and $BC = 2x = 8$.

Thus, the correct answer is **C**.

7. Josh writes the numbers $1, 2, 3, \ldots, 99, 100$. He marks out 1, skips the next number (2), marks out 3, and continues skipping and marking out the next number to the end of his list. Then he goes back to the start of his list, marks out the first remaining number (2), skips the next number (4), marks out 6, skips 8, marks out 10, and so on to the end. Josh continues in this manner until only one number remains. What is that number?

- A 13
- B 32
- C 56
- D 64
- E 96

Solution:

The first pass removes the odd numbers, leaving the multiples of 2. The second pass removes $2, 6, 10, \ldots,$ leaving the multiples of 4. In general, after the nth pass only the multiples of 2^n remain. The surviving number is the highest power of 2 not exceeding 100, which is $2^6 = 64$.

Thus, the correct answer is **D**.

8. A thin piece of wood of uniform density in the shape of an equilateral triangle with side length 3 inches weighs 12 ounces. A second piece of the same type of wood, with the same thickness, also in the shape of an equilateral triangle, has side length 5 inches. Which of the following is closest to the weight, in ounces, of the second piece?

- A 14.0
- B 16.0
- C 20.0
- D 33.3
- E 55.6

Solution:

Weight is proportional to area, and area scales with the square of the side length. The second side is $\frac{5}{3}$ times the first, so its weight is $12 \cdot \left(\frac{5}{3}\right)^2 = \frac{100}{3} \approx 33.3$ ounces.

Thus, the correct answer is **D**.

9. Carl decided to fence in his rectangular garden. He bought 20 fence posts, placed one on each of the four corners, and spaced out the rest evenly along the edges of the garden, leaving exactly 4 yards between neighboring posts. The longer side of his garden, including the corners, has twice as many posts as the shorter side, including the corners. What is the area, in square yards, of Carl's garden?

- A 256
- B 336
- C 384
- D 448
- E 512

Solution:

Let the shorter side have x posts, so the longer side has $2x$. Counting all posts and subtracting the four corners counted twice, $2x + 2(2x) - 4 = 20$, giving $x = 4$. The shorter side has 4 posts, or $(4 - 1) \cdot 4 = 12$ yards, and the longer side has 8 posts, or $(8 - 1) \cdot 4 = 28$ yards. The area is $12 \cdot 28 = 336$.

Thus, the correct answer is **B**.

10. A quadrilateral has vertices $P(a, b), Q(b, a), R(-a, -b)$, and $S(-b, -a)$, where a and b are integers with $a > b > 0$. The area of $PQRS$ is 16. What is $a + b$?

A 4

B 5

C 6

D 12

E 13

Solution:

The sides $\overline{PQ}$ and $\overline{RS}$ have slope -1, and $\overline{QR}$ and $\overline{PS}$ have slope 1, so $PQRS$ is a rectangle with sides $(a-b)\sqrt{2}$ and $(a+b)\sqrt{2}$. Its area is $2(a-b)(a+b) = 2(a^2 - b^2) = 16$, so $a^2 - b^2 = 8$. The only perfect squares differing by 8 are 9 and 1, giving $a = 3, b = 1$, and $a + b = 4$.

Thus, the correct answer is **A**.

11. How many squares whose sides are parallel to the axes and whose vertices have coordinates that are integers lie entirely within the region bounded by the line $y = \pi x$, the line $y = -0.1$, and the line $x = 5.1$?

A 30

B 41

C 45

D 50

E 57

Solution:

A unit square in the strip $k \leq x \leq k + 1$ fits below $y = \pi x$ up to height $\lfloor \pi k \rfloor$. Counting 1×1 squares in the strips $1 \leq x \leq 5$ gives $3 + 6 + 9 + 12 = 30$. The 2×2 squares give $2 + 5 + 8 = 15$, and the 3×3 squares give $1 + 4 = 5$. There are no larger squares, so the total is $30 + 15 + 5 = 50$.

Thus, the correct answer is **D**.

12. All the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ are written in a 3×3 array of squares, one number in each square, in such a way that if two numbers are consecutive then they occupy squares that share an edge. The numbers in the four corners add up to 18. What number is in the center?

- A 5
- B 6
- C 7
- D 8
- E 9

Solution:

Color the grid like a checkerboard so the four corners and the center share one color. Since consecutive numbers occupy adjacent (opposite colored) squares, the numbers alternate parity along the chain, so the five same-colored cells contain the five odd numbers $1, 3, 5, 7, 9$, which sum to 25. The four corners add to 18, so the center is $25 - 18 = 7$.

Thus, the correct answer is **C**.

13. Alice and Bob live 10 miles apart. One day Alice looks due north from her house and sees an airplane. At the same time Bob looks due west from his house and sees the same airplane. The angle of elevation of the airplane is 30° from Alice's position and 60° from Bob's position. Which of the following is closest to the airplane's altitude, in miles?

A 3.5

B 4

C 4.5

D 5

E 5.5

Solution:

Let the airplane be at C, directly above point D on the ground at altitude h. Triangles ACD and BCD are 30-60-90 right triangles, so $AD = \sqrt{3}\,h$ and $BD = \frac{h}{\sqrt{3}}$. Since Alice looks north and Bob looks west, $\angle ADB = 90^\circ$, so $AD^2 + BD^2 = AB^2 = 100$. Then $3h^2 + \frac{h^2}{3} = \frac{10h^2}{3} = 100$, giving $h = \sqrt{30} \approx 5.48$, closest to 5.5.

Thus, the correct answer is **E**.

14. The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

A $\dfrac{1+\sqrt{5}}{2}$

B 2

C $\sqrt{5}$

D 3

E 4

Solution:

Let r be the common ratio. Since the second term is 1, the first term is $\dfrac{1}{r}$, so $S = \dfrac{1/r}{1-r} = \dfrac{1}{r-r^2}$. Because $S > 0$, it is smallest when $r - r^2$ is largest. The parabola $r - r^2$ peaks at $r = \frac{1}{2}$, where it equals $\frac{1}{4}$, so the smallest value of S is $\dfrac{1}{1/4} = 4.$

Thus, the correct answer is **E**.

15. All the numbers $2, 3, 4, 5, 6, 7$ are assigned to the six faces of a cube, one number to each face. For each of the eight vertices of the cube, a product of three numbers is computed, where the three numbers are the numbers assigned to the three faces that include that vertex. What is the greatest possible value of the sum of these eight products?

- A 312
- B 343
- C 625
- D 729
- E 1680

Solution:

Pair the opposite faces as $(a, b), (c, d), (e, f)$. Each vertex product uses one face from each pair, so the sum of all eight products factors as $(a + b)(c + d)(e + f)$. The three factors have fixed total $2 + 3 + 4 + 5 + 6 + 7 = 27$, and a product with fixed sum is largest when the factors are equal, at 9 each. This balance is achievable with $(2, 7), (3, 6), (4, 5)$, giving $9 \cdot 9 \cdot 9 = 729$.

Thus, the correct answer is **D**.

16. In how many ways can 345 be written as the sum of an increasing sequence of two or more consecutive positive integers?

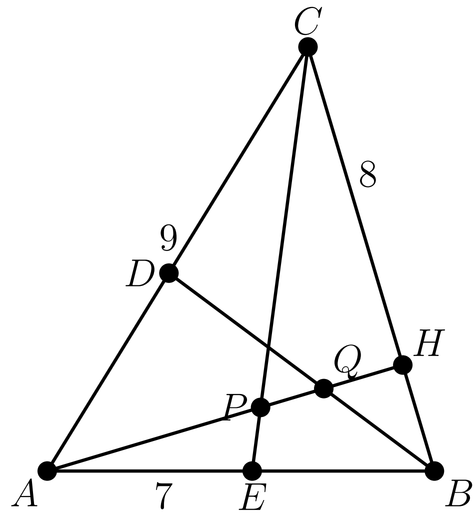
- A 1
- B 3
- C 5
- D 6
- E 7

Solution:

A sum of consecutive integers equals the count times the median. For an odd number of terms, the median is an integer divisor of $345,$ giving runs of 3 (median 115), 5 (median 69), 15 (median 23), and 23 (median 15) terms. For an even number of terms $2k,$ the median is a half-integer, giving runs of $2, 6,$ and 10 terms. Longer runs would force negative terms. This gives $4 + 3 = 7$ ways.

Thus, the correct answer is **E**.

17. In $\triangle ABC$ shown in the figure, $AB = 7, BC = 8, CA = 9,$ and $\overline{AH}$ is an altitude. Points D and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $\overline{BD}$ and $\overline{CE}$ are angle bisectors, intersecting $\overline{AH}$ at Q and P, respectively. What is PQ?



A 1

B $\frac{5}{8}\sqrt{3}$

C $\frac{4}{5}\sqrt{2}$

D $\frac{8}{15}\sqrt{5}$

E $\frac{6}{5}$

Solution:

Let $x = BH$. Then $CH = 8 - x$, and from the two right triangles $AH^2 = 7^2 - x^2 = 9^2 - (8-x)^2$. This gives $x = 2$ and $AH = \sqrt{45}$. By the angle bisector theorem in $\triangle ACH$, $\frac{AP}{PH} = \frac{CA}{CH} = \frac{9}{6}$, so $AP = \frac{3}{5}AH$. Similarly in $\triangle ABH$, $\frac{AQ}{QH} = \frac{BA}{BH} = \frac{7}{2}$, so $AQ = \frac{7}{9}AH$. Then $PQ = AQ - AP = \left(\frac{7}{9} - \frac{3}{5}\right) AH = \frac{8}{45}\sqrt{45} = \frac{8}{15}\sqrt{5}$.

Thus, the correct answer is **D**.

18. What is the area of the region enclosed by the graph of the equation $x^2 + y^2 = |x| + |y|$?

- A $\pi + \sqrt{2}$
- B $\pi + 2$
- C $\pi + 2\sqrt{2}$
- D $2\pi + \sqrt{2}$
- E $2\pi + 2\sqrt{2}$

Solution:

By symmetry, consider the first quadrant, where the equation is $x^2 + y^2 = x + y$, or $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}$. This is a circle centered at $\left(\frac{1}{2}, \frac{1}{2}\right)$ passing through $(1, 0)$ and $(0, 1)$; since the center is the midpoint of that chord, the enclosed first-quadrant region is the right triangle with legs to $(1, 0)$ and $(0, 1)$ (area $\frac{1}{2}$) plus a semicircle of radius $\dfrac{\sqrt{2}}{2}$ (area $\dfrac{\pi}{4}$). Multiplying by 4 for all quadrants gives $4\left(\frac{1}{2} + \frac{\pi}{4}\right) = \pi + 2.$

Thus, the correct answer is **B**.

19. Tom, Dick, and Harry are playing a game. Starting at the same time, each of them flips a fair coin repeatedly until he gets his first head, at which point he stops. What is the probability that all three flip their coins the same number of times?

A $\frac{1}{8}$

B $\frac{1}{7}$

C $\frac{1}{6}$

D $\frac{1}{4}$

E $\frac{1}{3}$

Solution:

A player's first head comes on flip n with probability $\left(\frac{1}{2}\right)^n$. All three stopping on the same flip n has probability $\left(\left(\frac{1}{2}\right)^n\right)^3 = \left(\frac{1}{8}\right)^n$. Summing over $n \geq 1$, $\sum_{n=1}^{\infty} \left(\frac{1}{8}\right)^n = \frac{1/8}{1-1/8} = \frac{1}{7}$.

Thus, the correct answer is **B**.

20. A set of teams held a round-robin tournament in which every team played every other team exactly once. Every team won 10 games and lost 10 games; there were no ties. How many sets of three teams $\{A, B, C\}$ were there in which A beat B, B beat C, and C beat A?

A 385

B 665

C 945

D 1140

E 1330

Solution:

Since each team won 10 and lost $10,$ there are 21 teams and $\binom{21}{3} = 1330$ triples. A triple is not cyclic exactly when one team beats both others. Choosing that team (21 ways) and 2 of the 10 teams it beat gives $21 \cdot \binom{10}{2} = 21 \cdot 45 = 945$ non-cyclic triples. Thus the cyclic triples number $1330 - 945 = 385.$

Thus, the correct answer is **A**.

21. Let $ABCD$ be a unit square. Let Q_1 be the midpoint of $\overline{CD}$. For $i = 1, 2, \ldots,$ let P_i be the intersection of $\overline{AQ_i}$ and $\overline{BD}$, and let Q_{i+1} be the foot of the perpendicular from P_i to $\overline{CD}$. What is

$$\sum_{i=1}^{\infty} \text{Area of } \triangle DQ_iP_i?$$

A $\frac{1}{6}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E 1

Solution:

Place $D = (0,0), C = (1,0), B = (1,1), A = (0,1)$, and let $q_i = DQ_i$. Intersecting line AQ_i with $\overline{BD}$ (the line $y = x$) gives P_i with both coordinates $\frac{q_i}{1+q_i}$, so $q_{i+1} = \frac{q_i}{1+q_i}$. From $q_1 = \frac{1}{2}$ this yields $q_i = \frac{1}{i+1}$. The base of $\triangle DQ_iP_i$ is $DQ_i = \frac{1}{i+1}$ and its height is the y-coordinate of P_i, which is $q_{i+1} = \frac{1}{i+2}$. Then

$$\text{Area of } \triangle DQ_iP_i = \tfrac{1}{2} \cdot \frac{1}{i+1} \cdot \frac{1}{i+2} = \tfrac{1}{2}\left(\frac{1}{i+1} - \frac{1}{i+2}\right).$$

Summing telescopes to $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

Thus, the correct answer is **B**.

22. For a certain positive integer n less than $1000,$ the decimal equivalent of $\frac{1}{n}$ is $0.\overline{abcdef}$, a repeating decimal of period $6,$ and the decimal equivalent of $\frac{1}{n+6}$ is $0.\overline{wxyz}$, a repeating decimal of period $4.$ In which interval does n lie?

A $[1, 200]$

B $[201, 400]$

C $[401, 600]$

D $[601, 800]$

E $[801, 999]$

Solution:

Period 6 requires $n \mid 10^6 - 1 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37.$ Period 4 requires $n+6 \mid 10^4 - 1 = 3^2 \cdot 11 \cdot 101$ but $n+6 \nmid 10^2 - 1 = 3^2 \cdot 11$ (else the period would be 1 or 2). Hence $101 \mid n+6,$ so $n = 101k - 6.$ For $n < 1000, k \in \{1, 3, 9\},$ giving $n \in \{95, 297, 903\}.$ Only $297 = 3^3 \cdot 11$ divides $10^6 - 1,$ so $n = 297,$ which lies in $[201, 400].$

Thus, the correct answer is **B**.

23. What is the volume of the region in three-dimensional space defined by the inequalities $|x| + |y| + |z| \leq 1$ and $|x| + |y| + |z - 1| \leq 1$?

A $\frac{1}{6}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E 1

Solution:

The region $|x| + |y| + |z| \leq 1$ is a regular octahedron with vertices at $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$, whose volume is $2 \cdot \frac{1}{3} \cdot (\sqrt{2})^2 \cdot 1 = \frac{4}{3}$. The second region is the same octahedron shifted up by 1. Their intersection is bounded by another regular octahedron with diagonals of length 1, half the linear dimensions of the first, so its volume is $\left(\frac{1}{2}\right)^3 \cdot \frac{4}{3} = \frac{1}{6}$.

Thus, the correct answer is **A**.

24. There are exactly $77{,}000$ ordered quadruples (a, b, c, d) such that $\gcd(a, b, c, d) = 77$ and $\operatorname{lcm}(a, b, c, d) = n$. What is the smallest possible value of n?

A $13{,}860$

B $20{,}790$

C $21{,}560$

D $27{,}720$

E $41{,}580$

Solution:

Writing each entry as 77 times a reduced value, we need $\gcd = 1$ and $\operatorname{lcm} = m = n/77$. For each prime p dividing m with maximum exponent M, the number of valid exponent quadruples is $(M+1)^4 - 2M^4 + (M-1)^4 = 2(6M^2+1)$. The total over all primes must equal $77{,}000 = 2^3 \cdot 5^3 \cdot 7 \cdot 11$. Since $2(6M^2+1)$ equals $14, 50,$ and 110 for $M = 1, 2, 3,$ and $14 \cdot 50 \cdot 110 = 77{,}000$, exactly three primes divide m, with maximum exponents $1, 2, 3$. To minimize $m = n/77$, assign the largest exponent to the smallest prime: $m = 2^3 \cdot 3^2 \cdot 5 = 360$, so $n = 77 \cdot 360 = 27{,}720.$

Thus, the correct answer is **D**.

25. The sequence (a_n) is defined recursively by $a_0 = 1, a_1 = \sqrt[19]{2}$, and $a_n = a_{n-1}a_{n-2}^2$ for $n \geq 2$. What is the smallest positive integer k such that the product $a_1 a_2 \cdots a_k$ is an integer?

A 17

B 18

C 19

D 20

E 21

Solution:

Write $a_n = 2^{b_n/19}$. The recursion becomes $b_0 = 0, b_1 = 1, b_n = b_{n-1} + 2b_{n-2}$, solved by $b_n = \frac{1}{3}\left(2^n - (-1)^n\right)$. The product $a_1 \cdots a_k$ is an integer exactly when $19 \mid b_1 + \cdots + b_k$. For odd k, this sum equals $\frac{1}{3}(2^{k+1} - 1)$, which is a multiple of 19 exactly when $19 \mid 2^{k+1} - 1$, i.e. when the order of 2 modulo 19, namely 18, divides $k + 1$. The smallest such odd k is $k + 1 = 18$, so $k = 17$.

Thus, the correct answer is **A**.

Problems: https://live.poshenloh.com/past-contests/amc12/2016B