2016 AMC 12A Solutions

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1. What is the value of

$$\frac{11! - 10!}{9!}?$$

A 99

B 100

C 110

D 121

E 132

Solution:

Factoring the numerator gives

$$\frac{11! - 10!}{9!} = \frac{10!(11-1)}{9!} = \frac{10 \cdot 9! \cdot 10}{9!} = 100.$$

Thus, the correct answer is **B**.

2. For what value of x does $10^x \cdot 100^{2x} = 1000^5$?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

Since $100 = 10^2$ and $1000 = 10^3$, the equation becomes

$$10^x \cdot 10^{4x} = 10^{15},$$

so $10^{5x} = 10^{15}$. Then $5x = 15$, giving $x = 3$.

Thus, the correct answer is **C**.

3. The remainder function can be defined for all real numbers x and y with $y \neq 0$ by

$$\text{rem}(x, y) = x - y\left\lfloor \frac{x}{y} \right\rfloor,$$

where $\left\lfloor \frac{x}{y} \right\rfloor$ denotes the greatest integer less than or equal to $\frac{x}{y}$. What is the value of $\text{rem}\left(\frac{3}{8}, -\frac{2}{5}\right)$?

A $-\frac{3}{8}$

B $-\frac{1}{40}$

C 0

D $\frac{3}{8}$

E $\frac{31}{40}$

Solution:

First,

$$\frac{x}{y} = \frac{3/8}{-2/5} = \frac{3}{8} \cdot \left(-\frac{5}{2}\right) = -\frac{15}{16},$$

and $\left\lfloor -\frac{15}{16} \right\rfloor = -1.$

Therefore

$$\text{rem}\left(\frac{3}{8}, -\frac{2}{5}\right) = \frac{3}{8} - \left(-\frac{2}{5}\right)(-1) = \frac{3}{8} - \frac{2}{5} = \frac{15 - 16}{40} = -\frac{1}{40}.$$

Thus, the correct answer is **B**.

4. The mean, median, and mode of the 7 data values $60, 100, x, 40, 50, 200, 90$ are all equal to x. What is the value of x?

A 50

B 60

C 75

D 90

E 100

Solution:

The mean condition gives

$$\frac{60+100+x+40+50+200+90}{7} = \frac{540+x}{7} = x,$$

so $540 + x = 7x$ and $x = 90.$

In nondecreasing order the data are $40, 50, 60, 90, 90, 100, 200,$ so the median is 90 and the mode is $90,$ as required.

Thus, the correct answer is **D**.

5. Goldbach's conjecture states that every even integer greater than 2 can be written as the sum of two prime numbers (for example, $2016 = 13 + 2003$). So far, no one has been able to prove that the conjecture is true, and no one has found a counterexample to show that the conjecture is false. What would a counterexample consist of?

A. an odd integer greater than 2 that can be written as the sum of two prime numbers

B. an odd integer greater than 2 that cannot be written as the sum of two prime numbers

C. an even integer greater than 2 that can be written as the sum of two numbers that are not prime

D. an even integer greater than 2 that can be written as the sum of two prime numbers

E. an even integer greater than 2 that cannot be written as the sum of two prime numbers

Solution:

A counterexample must satisfy the hypothesis of being an even integer greater than 2 while failing the conclusion that it can be written as the sum of two prime numbers.

Thus, the correct answer is **E**.

6. A triangular array of 2016 coins has 1 coin in the first row, 2 coins in the second row, 3 coins in the third row, and so on up to N coins in the Nth row. What is the sum of the digits of N?

- A 6
- B 7
- C 8
- D 9
- E 10

Solution:

The total number of coins is

$$1 + 2 + \cdots + N = \frac{N(N+1)}{2} = 2016,$$

so $N(N+1) = 4032.$ Since $63 \cdot 64 = 4032,$ we have $N = 63,$ and the sum of its digits is $6 + 3 = 9.$

Thus, the correct answer is **D**.

7. Which of these describes the graph of $x^2(x+y+1) = y^2(x+y+1)$?

A two parallel lines

B two intersecting lines

C three lines that all pass through a common point

D three lines that do not all pass through a common point

E a line and a parabola

Solution:

Moving all terms to one side gives

$$(x^2 - y^2)(x+y+1) = 0,$$

which factors as

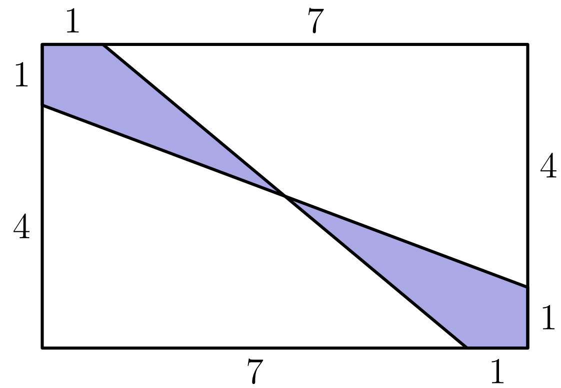
$$(x-y)(x+y)(x+y+1) = 0.$$

The graph is therefore the union of the lines $x = y$, $x = -y$, and $x+y+1 = 0$.

The first two lines intersect at the origin, but the third line $x + y = -1$ is parallel to $x + y = 0$ and does not pass through the origin. So the graph consists of three lines that do not all pass through a common point.

Thus, the correct answer is **D**.

8. What is the area of the shaded region of the given 8×5 rectangle?



A $4\frac{3}{4}$

B 5

C $5\frac{1}{4}$

D $6\frac{1}{2}$

E 8

Solution:

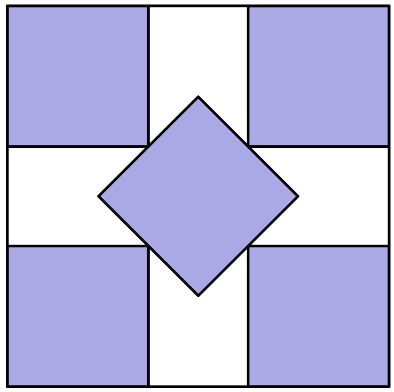
The diagonal of the rectangle from the upper-left corner to the lower-right corner divides the shaded region into four triangles, all meeting at the center of the rectangle.

Two of these triangles have a horizontal base of length 1 and altitude $\frac{1}{2} \cdot 5 = \frac{5}{2}$, and the other two have a vertical base of length 1 and altitude $\frac{1}{2} \cdot 8 = 4$. The total area is

$$2 \cdot \frac{1}{2} \cdot 1 \cdot \frac{5}{2} + 2 \cdot \frac{1}{2} \cdot 1 \cdot 4 = \frac{5}{2} + 4 = \frac{13}{2}.$$

Thus, the correct answer is **D**.

9. The five small shaded squares inside this unit square are congruent and have disjoint interiors. The midpoint of each side of the middle square coincides with one of the vertices of the other four small squares as shown. The common side length is $\dfrac{a-\sqrt{2}}{b}$, where a and b are positive integers. What is $a+b$?



A 7

B 8

C 9

D 10

E 11

Solution:

Let x be the common side length. The diagonal of the unit square has length $\sqrt{2}$ and consists of two small-square diagonals (each $x\sqrt{2}$) plus one small-square side length x, so

$$2x\sqrt{2}+x=\sqrt{2}.$$

Solving,

$$x=\frac{\sqrt{2}}{2\sqrt{2}+1}=\frac{\sqrt{2}\,(2\sqrt{2}-1)}{(2\sqrt{2}+1)(2\sqrt{2}-1)}=\frac{4-\sqrt{2}}{7}.$$

Thus $a=4, b=7,$ and $a+b=11.$

Thus, the correct answer is **E**.

10. Five friends sat in a movie theater in a row containing 5 seats, numbered 1 to 5 from left to right. (The directions "left" and "right" are from the point of view of the people as they sit in the seats.) During the movie Ada went to the lobby to get some popcorn. When she returned, she found that Bea had moved two seats to the right, Ceci had moved one seat to the left, and Dee and Edie had switched seats, leaving an end seat for Ada. In which seat had Ada been sitting before she got up?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

The net displacement of all five friends is zero. Dee and Edie swapped seats, so their movements cancel. Bea moved $+2$ and Ceci moved -1, a net of $+1$, so Ada must move -1 to balance.

Ada returns to an end seat; since she moved one seat to the left, that seat must be seat 1, so she had been sitting in seat 2.

Thus, the correct answer is **B**.

11. Each of the 100 students in a certain summer camp can either sing, dance, or act. Some students have more than one talent, but no student has all three talents. There are 42 students who cannot sing, 65 students who cannot dance, and 29 students who cannot act. How many students have two of these talents?

A 16

B 25

C 36

D 49

E 64

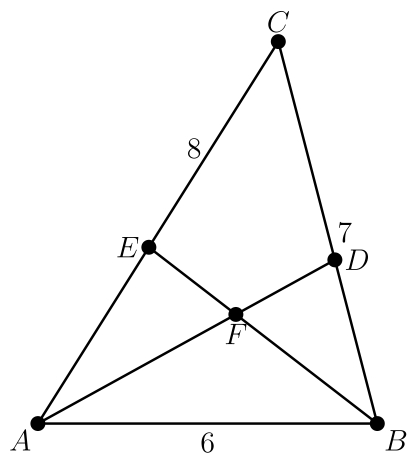
Solution:

The numbers who can sing, dance, and act are $100 - 42 = 58, 100 - 65 = 35,$ and $100 - 29 = 71,$ respectively, for a total of $58 + 35 + 71 = 164.$

Since no student has all three talents, each student has one or two talents, so single-talent students are counted once and two-talent students are counted twice. The number counted twice is $164 - 100 = 64.$

Thus, the correct answer is **E**.

12. In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point D lies on $\overline{BC}$, and $\overline{AD}$ bisects $\angle BAC$. Point E lies on $\overline{AC}$, and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F. What is the ratio $AF : FD$?



A $3 : 2$

B $5 : 3$

C $2 : 1$

D $7 : 3$

E $5 : 2$

Solution:

Applying the Angle Bisector Theorem to $\triangle ABC$ gives $BD : DC = AB : AC = 6 : 8$, so $BD = \dfrac{6}{6+8} \cdot 7 = 3$.

Now BF lies along the bisector of $\angle ABD$ in $\triangle ABD$, so by the Angle Bisector Theorem again,

$$AF : FD = AB : BD = 6 : 3 = 2 : 1.$$

Thus, the correct answer is **C**.

13. Let N be a positive multiple of 5. One red ball and N green balls are arranged in a line in random order. Let $P(N)$ be the probability that at least $\frac{3}{5}$ of the green balls are on the same side of the red ball. Observe that $P(5) = 1$ and that $P(N)$ approaches $\frac{4}{5}$ as N grows large. What is the sum of the digits of the least value of N such that $P(N) < \frac{321}{400}$?

A 12

B 14

C 16

D 18

E 20

Solution:

Write $N = 5k$. Number the positions of the red ball $0, 1, \ldots, 5k$ from one end; there are $5k + 1$ equally likely positions.

Fewer than $\frac{3}{5}$ of the green balls lie on each side exactly when the red ball is in one of the positions $2k + 1, 2k + 2, \ldots, 3k - 1$, which is $k - 1$ positions. Hence

$$P(N) = 1 - \frac{k-1}{5k+1} = \frac{4k+2}{5k+1}.$$

Solving $\frac{4k+2}{5k+1} < \frac{321}{400}$ gives $400(4k + 2) < 321(5k + 1)$, so $1600k + 800 < 1605k + 321$ and $5k > 479$, meaning $k > 95.8$. Thus $k = 96$ and $N = 480$, whose digit sum is $4 + 8 + 0 = 12$.

Thus, the correct answer is **A**.

14. Each vertex of a cube is to be labeled with an integer from 1 through 8, with each integer being used once, in such a way that the sum of the four numbers on the vertices of a face is the same for each face. Arrangements that can be obtained from each other through rotations of the cube are considered to be the same. How many different arrangements are possible?

- A 1
- B 3
- C 6
- D 12
- E 24

Solution:

Each vertex belongs to 3 faces, so $6S = 3(1+2+\cdots+8) = 108$, giving each face-sum $S = 18$.

The four-element subsets containing 1 with sum 18 are $\{1,2,7,8\}, \{1,3,6,8\}, \{1,4,5,8\}$, and $\{1,4,6,7\}$. Three of these contain both 1 and 8, so 1 and 8 must lie on two adjacent vertices.

Rotate the cube so that 1 is at the lower-left-front vertex and 8 at the lower-right-front vertex. The numbers $4, 6, 7$ must label the remaining vertices of the face containing 1, which can be done in $3! = 6$ ways; then $5, 3, 2$ are forced onto the opposite vertices. Hence there are 6 arrangements.

Thus, the correct answer is **C**.

15. Circles with centers $P, Q,$ and $R,$ having radii $1, 2,$ and $3,$ respectively, lie on the same side of line l and are tangent to l at P', Q', and R', respectively, with Q' between P' and R'. The circle with center Q is externally tangent to each of the other two circles. What is the area of $\triangle PQR$?

A 0

B $\sqrt{\frac{2}{3}}$

C 1

D $\sqrt{6}-\sqrt{2}$

E $\sqrt{\frac{3}{2}}$

Solution:

The centers lie at heights $1, 2,$ and 3 above line l. Since circle Q is externally tangent to circle P, we have $PQ = 3,$ so the horizontal distance is $P'Q' = \sqrt{3^2 - 1^2} = \sqrt{8}.$ Since circle Q is tangent to circle R, we have $QR = 5,$ so $Q'R' = \sqrt{5^2 - 1^2} = \sqrt{24}.$

Place $P = (0, 1), Q = (\sqrt{8}, 2),$ and $R = (\sqrt{8} + \sqrt{24}, 3)$. By the shoelace formula, the area is

$$\frac{1}{2}\left|\sqrt{8}(3-1) + (\sqrt{8}+\sqrt{24})(1-2)\right| = \frac{1}{2}\left(\sqrt{24}-\sqrt{8}\right) = \sqrt{6}-\sqrt{2}.$$

Thus, the correct answer is **D**.

16. The graphs of $y = \log_3 x, y = \log_x 3, y = \log_{1/3} x$, and $y = \log_x \frac{1}{3}$ are plotted on the same set of axes. How many points in the plane with positive x-coordinates lie on two or more of the graphs?

A 2

B 3

C 4

D 5

E 6

Solution:

Let $u = \log_3 x$. Then $\log_x 3 = \frac{1}{u}, \log_{1/3} x = -u$, and $\log_x \frac{1}{3} = -\frac{1}{u}$. Two graphs meet where two of $u, \frac{1}{u}, -u, -\frac{1}{u}$ are equal for some valid $x > 0$.

Setting $u = \frac{1}{u}$ gives $u = \pm 1$, so $x = 3$ or $x = \frac{1}{3}$; setting $-u = -\frac{1}{u}$ gives the same values. Setting $u = -u$ gives $u = 0$, i.e. $x = 1$, where $\log_3 x$ and $\log_{1/3} x$ are both 0. The remaining pairings have no real solution.

The distinct intersection points are $(1, 0), (3, 1), \left(\frac{1}{3}, -1\right), (3, -1)$, and $\left(\frac{1}{3}, 1\right)$, so there are 5.

Thus, the correct answer is **D**.

17. Let $ABCD$ be a square. Let $E, F, G,$ and H be the centers, respectively, of equilateral triangles with bases $\overline{AB}, \overline{BC}, \overline{CD},$ and $\overline{DA},$ each exterior to the square. What is the ratio of the area of square $EFGH$ to the area of square $ABCD$?

A 1

B $\dfrac{2+\sqrt{3}}{3}$

C $\sqrt{2}$

D $\dfrac{\sqrt{2}+\sqrt{3}}{2}$

E $\sqrt{3}$

Solution:

Let square $ABCD$ have side length $6.$ Each equilateral triangle has height $3\sqrt{3},$ and its center lies $\frac{1}{3}$ of that height, namely $\sqrt{3},$ from the square's side.

Square $ABCD$ has diagonal $6\sqrt{2}.$ Square $EFGH$ has diagonal equal to the side of $ABCD$ plus twice $\sqrt{3},$ namely $6+2\sqrt{3}.$ The area ratio is the square of the ratio of diagonals:

$$\left(\frac{6+2\sqrt{3}}{6\sqrt{2}}\right)^2=\left(\frac{3+\sqrt{3}}{3\sqrt{2}}\right)^2=\frac{12+6\sqrt{3}}{18}=\frac{2+\sqrt{3}}{3}.$$

Thus, the correct answer is **B**.

18. For some positive integer n, the number $110n^3$ has 110 positive integer divisors, including 1 and the number $110n^3$. How many positive integer divisors does the number $81n^4$ have?

- A 110
- B 191
- C 261
- D 325
- E 425

Solution:

Write $110n^3 = p_1^{r_1}p_2^{r_2}\cdots$ so that the number of divisors is $(r_1+1)(r_2+1)\cdots = 110$. Since $110 = 2\cdot 5\cdot 11$, there are exactly three distinct primes, which must be $2, 5, 11$, with exponents $1, 4, 10$ in some order.

Taking $r_1 = 1, r_2 = 4, r_3 = 10$ for the primes $2, 5, 11$ gives

$$n^3 = \frac{2^1\cdot 5^4\cdot 11^{10}}{2\cdot 5\cdot 11} = 5^3\cdot 11^9, \quad \text{so} \quad n = 5\cdot 11^3.$$

Then $81n^4 = 3^4\cdot 5^4\cdot 11^{12}$, and since $3, 5, 11$ are distinct primes, the number of divisors is

$$(4+1)(4+1)(12+1) = 5\cdot 5\cdot 13 = 325.$$

Thus, the correct answer is **D**.

19. Jerry starts at 0 on the real number line. He tosses a fair coin 8 times. When he gets heads, he moves 1 unit in the positive direction; when he gets tails, he moves 1 unit in the negative direction. The probability that he reaches 4 at some time during this process is $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a+b$? (For example, he succeeds if his sequence of tosses is HTHHHHHH.)

- A 69
- B 151
- C 257
- D 293
- E 313

Solution:

Count the sequences of 8 tosses whose running total reaches $4.$ With at most 2 tails he certainly reaches $4,$ contributing

$$\binom{8}{0}+\binom{8}{1}+\binom{8}{2}=1+8+28=37$$

sequences.

With exactly 3 tails he reaches 4 only if he does so before the second tail, which allows at most one tail in the first 5 tosses; this gives $4+4=8$ sequences. With exactly 4 tails, only HHHHTTTT works, giving $1.$ He cannot reach 4 with fewer than 4 heads.

So there are $37+8+1=46$ favorable sequences out of $2^8=256,$ a probability of $\frac{46}{256}=\frac{23}{128}$. Then $a+b=23+128=151.$

Thus, the correct answer is **B**.

20. A binary operation $\diamond$ has the properties that $a \diamond (b \diamond c) = (a \diamond b) \cdot c$ and that $a \diamond a = 1$ for all nonzero real numbers $a, b,$ and c. (Here the dot $\cdot$ represents the usual multiplication operation.) The solution to the equation $2016 \diamond (6 \diamond x) = 100$ can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p + q$?

A 109

B 201

C 301

D 3049

E $33{,}601$

Solution:

Setting $b = c = a$ gives $a \diamond 1 = a \diamond (a \diamond a) = (a \diamond a) \cdot a = a$. Then setting $c = b$ gives $a = a \diamond 1 = a \diamond (b \diamond b) = (a \diamond b) \cdot b$, so $a \diamond b = \frac{a}{b}$.

Therefore

$$2016 \diamond (6 \diamond x) = 2016 \diamond \frac{6}{x} = \frac{2016}{6/x} = 336x = 100,$$

so $x = \frac{100}{336} = \frac{25}{84}$ and $p + q = 25 + 84 = 109$.

Thus, the correct answer is **A**.

21. A quadrilateral is inscribed in a circle of radius $200\sqrt{2}$. Three of the sides of this quadrilateral have length 200. What is the length of its fourth side?

A 200

B $200\sqrt{2}$

C $200\sqrt{3}$

D $300\sqrt{2}$

E 500

Solution:

Let θ be the central angle subtending a side of length 200, with radius $R = 200\sqrt{2}$. By the law of cosines on the isosceles triangle from the center,

$$200^2 = 2R^2(1 - \cos\theta) = 160000(1 - \cos\theta),$$

so $\cos\theta = \dfrac{3}{4}$.

The fourth side subtends the central angle 3θ, and

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4 \cdot \frac{27}{64} - \frac{9}{4} = -\frac{9}{16}.$$

Its length squared is

$$2R^2(1 - \cos 3\theta) = 160000\left(1 + \frac{9}{16}\right) = 160000 \cdot \frac{25}{16} = 250000,$$

so the fourth side is 500.

Thus, the correct answer is **E**.

22. How many ordered triples (x, y, z) of positive integers satisfy $\operatorname{lcm}(x, y) = 72$, $\operatorname{lcm}(x, z) = 600$, and $\operatorname{lcm}(y, z) = 900$?

A 15

B 16

C 24

D 27

E 64

Solution:

Because $\operatorname{lcm}(x, y) = 2^3 \cdot 3^2$ and $\operatorname{lcm}(x, z) = 2^3 \cdot 3 \cdot 5^2$, the factor 5^2 divides z while neither x nor y is divisible by 5. Also 3^2 divides y, while neither x nor z is divisible by 3^2, and x must have the factor 2^3.

Writing $x = 2^3 \cdot 3^j, y = 2^k \cdot 3^2$, and $z = 2^m \cdot 3^n \cdot 5^2$, the lcm conditions require $\max(j, n) = 1$ and $\max(k, m) = 2$. There are 3 choices for (j, n) and 5 choices for (k, m), giving $3 \cdot 5 = 15$ ordered triples.

Thus, the correct answer is **A**.

23. Three numbers in the interval $[0, 1]$ are chosen independently and at random. What is the probability that the chosen numbers are the side lengths of a triangle with positive area?

A $\frac{1}{6}$

B $\frac{1}{3}$

C $\frac{1}{2}$

D $\frac{2}{3}$

E $\frac{5}{6}$

Solution:

The ordered triples (x, y, z) fill the unit cube of volume 1. They fail to form a triangle exactly when one value is at least the sum of the other two.

The region $z \geq x + y$ is a tetrahedron with vertices $(0,0,0), (0,0,1), (0,1,1), (1,0,1)$ of volume $\frac{1}{6}$. The analogous regions $x \geq y + z$ and $y \geq x + z$ also have volume $\frac{1}{6}$ and have disjoint interiors. So the failure probability is $3 \cdot \frac{1}{6} = \frac{1}{2}$, and the triangle probability is $1 - \frac{1}{2} = \frac{1}{2}$.

Thus, the correct answer is **C**.

24. There is a smallest positive real number a such that there exists a positive real number b such that all the roots of the polynomial $x^3 - ax^2 + bx - a$ are real. In fact, for this value of a the value of b is unique. What is this value of b?

A 8

B 9

C 10

D 11

E 12

Solution:

Since a and b are positive, all roots r, s, t must be positive. By Vieta's formulas, $r + s + t = a, rs + st + tr = b,$ and $rst = a,$ so $r + s + t = rst.$

By the AM-GM inequality, $27rst \leq (r + s + t)^3 = (rst)^3,$ so $a = rst \geq 3\sqrt{3},$ with equality if and only if $r = s = t = \sqrt{3}.$ At this smallest $a,$

$$b = rs + st + tr = 3r^2 = 3 \cdot 3 = 9.$$

Thus, the correct answer is **B**.

25. Let k be a positive integer. Bernardo and Silvia take turns writing and erasing numbers on a blackboard as follows: Bernardo starts by writing the smallest perfect square with $k+1$ digits. Every time Bernardo writes a number, Silvia erases the last k digits of it. Bernardo then writes the next perfect square, Silvia erases the last k digits of it, and this process continues until the last two numbers that remain on the board differ by at least 2. Let $f(k)$ be the smallest positive integer not written on the board. For example, if $k=1$, then the numbers that Bernardo writes are $16, 25, 36, 49,$ and 64, and the numbers showing on the board after Silvia erases are $1, 2, 3, 4,$ and 6, and thus $f(1)=5$. What is the sum of the digits of $f(2)+f(4)+f(6)+\cdots+f(2016)$?

- A 7986
- B 8002
- C 8030
- D 8048
- E 8064

Solution:

Take $k=2j$. The smallest perfect square with $k+1$ digits is $10^k=(10^j)^2$, and after Silvia erases, the numbers shown are $\lfloor n^2/10^k \rfloor$ for $n=10^j, 10^j+1, \ldots$ Consecutive terms increase by 0 or 1 until the first jump of at least 2.

That first jump occurs at $n=\dfrac{10^k}{2}+m$ with $m=10^j-1$, and one computes that the last number written before the gap gives

$$f(2j)=\frac{10^{2j}}{4}+10^j.$$

Summing over $j=1,\ldots,1008$,

$$\sum_{j=1}^{1008} f(2j)=25\sum_{j=0}^{1007}10^{2j}+10\sum_{j=0}^{1007}10^j=\underbrace{2525\cdots25}_{2016\text{ digits}}+\underbrace{111\cdots10}_{1009\text{ digits}}.$$

There are no carries, so the digit sum is $1008\cdot(2+5)+1008\cdot 1=1008\cdot 8=8064$.

Thus, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/2016A