2015 AMC 12B Solutions

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1. What is the value of $2-(-2)^{-2}$?

A -2

B $\frac{1}{16}$

C $\frac{7}{4}$

D $\frac{9}{4}$

E 6

Solution:

Since $(-2)^{-2}=\frac{1}{(-2)^2}=\frac{1}{4}$, we get $2-\frac{1}{4}=\frac{7}{4}$.

Thus, the correct answer is **C**.

2. Marie does three equally time-consuming tasks in a row without taking breaks. She begins the first task at 1:00 PM and finishes the second task at 2:40 PM. When does she finish the third task?

- A 3:10 PM
- B 3:30 PM
- C 4:00 PM
- D 4:10 PM
- E 4:30 PM

Solution:

The first two tasks together take 100 minutes, so each task takes 50 minutes.

The third task finishes 50 minutes after 2:40 PM, at 3:30 PM.

Thus, the correct answer is **B**.

3. Isaac has written down one integer two times and another integer three times. The sum of the five numbers is 100, and one of the numbers is 28. What is the other number?

A 8

B 11

C 14

D 15

E 18

Solution:

Write $2x + 3y = 100$. If 28 were written twice, then $3y = 100 - 56 = 44$, which is not a multiple of 3.

So 28 is written three times: $2x = 100 - 84 = 16$, giving $x = 8$.

Thus, the correct answer is **A**.

4. David, Hikmet, Jack, Marta, Rand, and Todd were in a 12-person race with 6 other people. Rand finished 6 places ahead of Hikmet. Marta finished 1 place behind Jack. David finished 2 places behind Hikmet. Jack finished 2 places behind Todd. Todd finished 1 place behind Rand. Marta finished in 6th place. Who finished in 8th place?

A David

B Hikmet

C Jack

D Rand

E Todd

Solution:

Marta is 6th, so Jack is 5th. Jack is 2 behind Todd, so Todd is 3rd. Todd is 1 behind Rand, so Rand is 2nd.

Rand is 6 ahead of Hikmet, so Hikmet is 8th. (David is 10th.)

Thus, the correct answer is **B**.

5. The Tigers beat the Sharks 2 out of the first 3 times they played. They then played N more times, and the Sharks ended up winning at least 95% of all the games played. What is the minimum possible value for N?

A 35

B 37

C 39

D 41

E 43

Solution:

The Sharks won 1 of the first 3 games. To reach 95% with the fewest extra games, they should win all N additional games, giving a win fraction $\dfrac{1+N}{3+N}$.

Requiring $\dfrac{1+N}{3+N} \geq \dfrac{19}{20}$ gives $20 + 20N \geq 57 + 19N$, so $N \geq 37$.

Thus, the correct answer is **B**.

6. Back in 1930, Tillie had to memorize her multiplication facts from 0×0 through 12×12. The multiplication table she was given had rows and columns labeled with the factors, and the products formed the body of the table. To the nearest hundredth, what fraction of the numbers in the body of the table are odd?

- A 0.21
- B 0.25
- C 0.46
- D 0.50
- E 0.75

Solution:

The body has $13 \times 13 = 169$ entries. A product is odd exactly when both factors are odd.

There are 6 odd numbers among $0, 1, \ldots, 12$, giving $6 \times 6 = 36$ odd entries. The fraction is $\dfrac{36}{169} = 0.213\ldots \approx 0.21.$

Thus, the correct answer is **A**.

7. A regular 15-gon has L lines of symmetry, and the smallest positive angle for which it has rotational symmetry is R degrees. What is $L + R$?

A 24

B 27

C 32

D 39

E 54

Solution:

A regular 15-gon has $L = 15$ lines of symmetry, and its smallest angle of rotational symmetry is $R = \frac{360}{15} = 24$ degrees.

Then $L + R = 15 + 24 = 39.$

Thus, the correct answer is **D**.

8. What is the value of $\left(625^{\log_5 2015}\right)^{\frac{1}{4}}$?

A 5

B $\sqrt[4]{2015}$

C 625

D 2015

E $\sqrt[4]{5^{2015}}$

Solution:

Since $625 = 5^4$, we have $625^{\log_5 2015} = 5^{4\log_5 2015} = \left(5^{\log_5 2015}\right)^4 = 2015^4$.

Taking the fourth root gives $\left(2015^4\right)^{1/4} = 2015$.

Thus, the correct answer is **D**.

9. Larry and Julius are playing a game, taking turns throwing a ball at a bottle sitting on a ledge. Larry throws first. The winner is the first person to knock the bottle off the ledge. At each turn the probability that a player knocks the bottle off the ledge is $\frac{1}{2}$, independently of what has happened before. What is the probability that Larry wins the game?

- A $\frac{1}{2}$
- B $\frac{3}{5}$
- C $\frac{2}{3}$
- D $\frac{3}{4}$
- E $\frac{4}{5}$

Solution:

Let x be the probability Larry wins. He wins right away with probability $\frac{1}{2}$, or both players miss (probability $\frac{1}{4}$) and the game restarts.

So $x = \frac{1}{2} + \frac{1}{4}x$, giving $\frac{3}{4}x = \frac{1}{2}$ and $x = \frac{2}{3}$.

Thus, the correct answer is **C**.

10. How many noncongruent integer-sided triangles with positive area and perimeter less than 15 are neither equilateral, isosceles, nor right triangles?

- A 3
- B 4
- C 5
- D 6
- E 7

Solution:

Let the distinct sides be $a < b < c$. Since $a + b > c$, the perimeter exceeds $2c$, so $2c < 15$ and $c \leq 6$.

The scalene triples with perimeter less than 15 are $(6, 5, 3)$, $(6, 5, 2)$, $(6, 4, 3)$, $(5, 4, 3)$, $(5, 4, 2)$, and $(4, 3, 2)$. Of these, only $(5, 4, 3)$ is a right triangle, leaving 5.

Thus, the correct answer is **C**.

11. The line $12x + 5y = 60$ forms a triangle with the coordinate axes. What is the sum of the lengths of the altitudes of this triangle?

A 20

B $\frac{360}{17}$

C $\frac{107}{5}$

D $\frac{43}{2}$

E $\frac{281}{13}$

Solution:

The line meets the axes at $(5, 0)$ and $(0, 12)$, so the triangle is right with legs 5 and 12 and hypotenuse 13. Its area is 30.

Two altitudes are the legs 5 and 12; the altitude to the hypotenuse is $\frac{2 \cdot 30}{13} = \frac{60}{13}$. The sum is $17 + \frac{60}{13} = \frac{281}{13}$.

Thus, the correct answer is **E**.

12. Let $a, b,$ and c be three distinct one-digit numbers. What is the maximum value of the sum of the roots of the equation $(x-a)(x-b)+(x-b)(x-c)=0$?

A 15

B 15.5

C 16

D 16.5

E 17

Solution:

Factoring gives $(x-b)\big(2x-(a+c)\big)=0$, so the roots are b and $\frac{a+c}{2}$. Their sum is $b+\frac{a+c}{2}$.

Using distinct digits, take $b=9$ and $a+c=8+7=15$, giving $9+7.5=16.5.$

Thus, the correct answer is **D**.

13. Quadrilateral $ABCD$ is inscribed in a circle with $\angle BAC = 70^\circ, \angle ADB = 40^\circ$, $AD = 4$, and $BC = 6$. What is AC?

A $3 + \sqrt{5}$

B 6

C $\frac{9}{2}\sqrt{2}$

D $8 - \sqrt{2}$

E 7

Solution:

Angles BAC and BDC subtend arc BC, so $\angle BDC = 70^\circ$. Then $\angle ADC = \angle ADB + \angle BDC = 110^\circ$.

Since $ABCD$ is cyclic, $\angle ABC = 180^\circ - 110^\circ = 70^\circ = \angle BAC$. Thus $\triangle ABC$ is isosceles with $AC = BC = 6$.

Thus, the correct answer is **B**.

14. A circle of radius 2 is centered at A. An equilateral triangle with side 4 has a vertex at A. What is the difference between the area of the region that lies inside the circle but outside the triangle and the area of the region that lies inside the triangle but outside the circle?

- A $8-\pi$
- B $\pi+2$
- C $2\pi-\frac{\sqrt{2}}{2}$
- D $4(\pi-\sqrt{3})$
- E $2\pi+\frac{\sqrt{3}}{2}$

Solution:

Let z be the area shared by the circle and triangle. The requested difference is $(\text{circle}-z)-(\text{triangle}-z)=\text{circle}-\text{triangle}.$

The circle has area $\pi\cdot 2^2=4\pi$, and the equilateral triangle has area $\frac{\sqrt{3}}{4}\cdot 4^2=4\sqrt{3}.$ The difference is $4\pi-4\sqrt{3}=4(\pi-\sqrt{3}).$

Thus, the correct answer is **D**.

15. At Rachelle's school an A counts 4 points, a B 3 points, a C 2 points, and a D 1 point. Her GPA on the four classes she is taking is computed as the total sum of points divided by 4. She is certain that she will get As in both Mathematics and Science, and at least a C in each of English and History. She thinks she has a $\frac{1}{6}$ chance of getting an A in English, and a $\frac{1}{4}$ chance of getting a B. In History, she has a $\frac{1}{4}$ chance of getting an A, and a $\frac{1}{3}$ chance of getting a B, independently of what she gets in English. What is the probability that Rachelle will get a GPA of at least 3.5?

A $\frac{11}{72}$

B $\frac{1}{6}$

C $\frac{3}{16}$

D $\frac{11}{24}$

E $\frac{1}{2}$

Solution:

Math and Science give 8 points, so Rachelle needs at least 6 more from English and History. The chance of a C is $1-\frac{1}{6}-\frac{1}{4}=\frac{7}{12}$ in English and $1-\frac{1}{4}-\frac{1}{3}=\frac{5}{12}$ in History.

Working over a denominator of $144: 8$ points has probability $\frac{1}{6}\cdot\frac{1}{4}=\frac{6}{144}$; 7 points has $\frac{1}{6}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{4}=\frac{17}{144}$; and 6 points has $\frac{1}{6}\cdot\frac{5}{12}+\frac{1}{4}\cdot\frac{1}{3}+\frac{7}{12}\cdot\frac{1}{4}=\frac{43}{144}$.

The total is $\frac{6+17+43}{144}=\frac{66}{144}=\frac{11}{24}$.

Thus, the correct answer is **D**.

16. A regular hexagon with sides of length 6 has an isosceles triangle attached to each side. Each of these triangles has two sides of length 8. The isosceles triangles are folded to make a pyramid with the hexagon as the base of the pyramid. What is the volume of the pyramid?

- A 18
- B 162
- C $36\sqrt{21}$
- D $18\sqrt{138}$
- E $54\sqrt{21}$

Solution:

The distance from the hexagon's center to a vertex is 6. A lateral edge has length 8, so the pyramid's height is $\sqrt{8^2 - 6^2} = \sqrt{28} = 2\sqrt{7}$.

The hexagon's area is $\dfrac{3\sqrt{3}}{2} \cdot 6^2 = 54\sqrt{3}$. Thus the volume is $\dfrac{1}{3} \cdot 54\sqrt{3} \cdot 2\sqrt{7} = 36\sqrt{21}$.

Thus, the correct answer is **C**.

17. An unfair coin lands on heads with a probability of $\frac{1}{4}$. When tossed n times, the probability of exactly two heads is the same as the probability of exactly three heads. What is the value of n?

A 5

B 8

C 10

D 11

E 13

Solution:

Setting the two probabilities equal and cancelling the common powers of $\frac{1}{4}$ and $\frac{3}{4}$ gives $\binom{n}{2} \cdot \frac{3}{4} = \binom{n}{3} \cdot \frac{1}{4}$.

This becomes $\frac{n(n-1)}{2} \cdot 3 = \frac{n(n-1)(n-2)}{6}$, so $\frac{3}{2} = \frac{n-2}{6}$, giving $n - 2 = 9$ and $n = 11$.

Thus, the correct answer is **D**.

18. For every composite positive integer n, define $r(n)$ to be the sum of the factors in the prime factorization of n. For example, $r(50) = 12$ because the prime factorization of 50 is $2 \cdot 5^2$, and $2 + 5 + 5 = 12$. What is the range of the function r, $\{r(n) : n \text{ is a composite positive integer}\}$?

A the set of positive integers

B the set of composite positive integers

C the set of even positive integers

D the set of integers greater than 3

E the set of integers greater than 4

Solution:

A composite number has at least two prime factors (with multiplicity), and the smallest prime is 2, so the least possible value is $2 + 2 = 4$.

Every integer greater than 3 is attained: $r(2^k) = 2k$ covers the even values ≥ 4, and $r(2^k \cdot 3) = 2k + 3$ covers the odd values ≥ 5. So the range is the integers greater than 3.

Thus, the correct answer is **D**.

19. In $\triangle ABC, \angle C = 90^\circ$ and $AB = 12$. Squares $ABXY$ and $ACWZ$ are constructed outside of the triangle. The points X, Y, Z, and W lie on a circle. What is the perimeter of the triangle?

A $12 + 9\sqrt{3}$

B $18 + 6\sqrt{3}$

C $12 + 12\sqrt{2}$

D 30

E 32

Solution:

The center O of the circle lies on the perpendicular bisectors of XY and ZW, which are the same as those of AB and AC. So O is the circumcenter of $\triangle ABC$, and since $\angle C = 90^\circ, O$ is the midpoint of AB.

Let $a = \frac{1}{2}BC$ and $b = \frac{1}{2}CA$. Then $a^2 + b^2 = 6^2$, and computing $OX^2 = OW^2$ gives $12^2 + 6^2 = b^2 + (a + 2b)^2$. Solving yields $a = b = 3\sqrt{2}$, so $BC = CA = 6\sqrt{2}$ and the perimeter is $12 + 12\sqrt{2}$.

Thus, the correct answer is **C**.

20. For every positive integer n, let $\text{mod}_5(n)$ be the remainder obtained when n is divided by 5. Define a function $f : \{0, 1, 2, 3, \ldots\} \times \{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3, 4\}$ recursively as follows:

$$f(i, j) = \begin{cases} \text{mod}_5(j + 1) & \text{if } i = 0 \text{ and } 0 \leq j \leq 4, \\ f(i - 1, 1) & \text{if } i \geq 1 \text{ and } j = 0, \text{ and} \\ f(i - 1, f(i, j - 1)) & \text{if } i \geq 1 \text{ and } 1 \leq j \leq 4. \end{cases}$$

What is $f(2015, 2)$?

A 0

B 1

C 2

D 3

E 4

Solution:

Computing $f(i, j)$ row by row from the definition, the column $j = 2$ stabilizes: $f(i, 2) = 1$ for all $i \geq 5$.

Since $2015 \geq 5$, we get $f(2015, 2) = 1$.

Thus, the correct answer is **B**.

21. Cozy the Cat and Dash the Dog are going up a staircase with a certain number of steps. However, instead of walking up the steps one at a time, both Cozy and Dash jump. Cozy goes two steps up with each jump (though if necessary, he will just jump the last step). Dash goes five steps up with each jump (though if necessary, he will just jump the last steps if there are fewer than 5 steps left). Suppose that Dash takes 19 fewer jumps than Cozy to reach the top of the staircase. Let s denote the sum of all possible numbers of steps this staircase can have. What is the sum of the digits of s?

A 9

B 11

C 12

D 13

E 15

Solution:

A staircase of t steps takes Cozy $\left\lceil \frac{t}{2} \right\rceil$ jumps and Dash $\left\lceil \frac{t}{5} \right\rceil$ jumps, and we need the difference to equal 19.

Checking the possibilities, the valid values are $t = 63, 64,$ and $66,$ so $s = 63 + 64 + 66 = 193.$ Its digit sum is $1 + 9 + 3 = 13.$

Thus, the correct answer is **D**.

22. Six chairs are evenly spaced around a circular table. One person is seated in each chair. Each person gets up and sits down in a chair that is not the same chair and is not adjacent to the chair he or she originally occupied, so that again one person is seated in each chair. In how many ways can this be done?

- A 14
- B 16
- C 18
- D 20
- E 24

Solution:

First imagine everyone moves to the chair directly opposite. The condition becomes: each person must sit in the same chair or an adjacent one. The number of people who keep their seat must be even (otherwise an odd-length gap cannot be filled).

If 0 keep their seat, everyone shifts left, shifts right, or swaps with a neighbor: 4 ways. If 2 keep their seats, those two are opposite or adjacent, giving $3+6=9$ ways, with the rest forced. If 4 keep their seats, there are 6 ways to choose them and the other two swap. If all 6 stay, 1 way. The total is $4+9+6+1=20.$

Thus, the correct answer is **D**.

23. A rectangular box measures $a \times b \times c$, where a, b, and c are integers and $1 \leq a \leq b \leq c$. The volume and the surface area of the box are numerically equal. How many ordered triples (a, b, c) are possible?

A 4

B 10

C 12

D 21

E 26

Solution:

Numerically equal volume and surface area means $abc = 2(ab + bc + ca)$. Rearranging shows $a \leq 6$, and $a = 1$ or $a = 2$ give no solutions.

For each remaining a, the equation factors: $a = 3$ gives $(b-6)(c-6) = 36$ with 5 solutions; $a = 4$ gives $(b-4)(c-4) = 16$ with 3 solutions; $a = 5$ gives $(3b - 10)(3c - 10) = 100$ with 1 valid solution; and $a = 6$ gives $(b-3)(c-3) = 9$ with 1 valid solution. That is $5 + 3 + 1 + 1 = 10$ triples.

Thus, the correct answer is **B**.

24. Four circles, no two of which are congruent, have centers at $A, B, C,$ and D, and points P and Q lie on all four circles. The radius of circle A is $\frac{5}{8}$ times the radius of circle B, and the radius of circle C is $\frac{5}{8}$ times the radius of circle D. Furthermore, $AB = CD = 39$ and $PQ = 48$. Let R be the midpoint of $\overline{PQ}$. What is $AR + BR + CR + DR$?

A 180

B 184

C 188

D 192

E 196

Solution:

Since every center is equidistant from P and Q, all four centers and R lie on the perpendicular bisector of PQ, with $PR = 24$. Suppose R lies between A and B. Let $y = AR$ and $x = \frac{1}{5}$ of circle A's radius. Then $y^2 + 24^2 = 25x^2$ and $(39 - y)^2 + 24^2 = 64x^2$. Subtracting gives $x^2 = 39 - 2y$, so $y^2 + 50y - 399 = 0$ and $y = 7$. Thus $AR = 7$ and $BR = 32$.

Because the circles are noncongruent, R does not lie between C and D. The analogous equations give $w^2 - 50w - 399 = 0$ with $w = CR = 57$, so $DR = 96$. The sum is $7 + 32 + 57 + 96 = 192$.

Thus, the correct answer is **D**.

25. A bee starts flying from point P_0. She flies 1 inch due east to point P_1. For $j \geq 1$, once the bee reaches point P_j, she turns 30° counterclockwise and then flies $j+1$ inches straight to point P_{j+1}. When the bee reaches P_{2015} she is exactly $a\sqrt{b}+c\sqrt{d}$ inches away from P_0, where $a, b, c,$ and d are positive integers and b and d are not divisible by the square of any prime. What is $a+b+c+d$?

- A 2016
- B 2024
- C 2032
- D 2040
- E 2048

Solution:

Place $P_0 = 0$ and let $z = e^{\pi i/6}$, so each step of length k in direction z^{k-1} gives $P_{2015} = \sum_{k=1}^{2015} kz^{k-1}$. Summing this (a differentiated geometric series) leads to $P_{2015} = \dfrac{1}{(z-1)^2}\left(2015z^{2016} - 2016z^{2015} + 1\right)$.

Since $z^{12} = 1$, we have $z^{2016} = 1$ and $z^{2015} = \frac{1}{z}$, so $P_{2015} = \dfrac{2016}{z(z-1)}$. Using $|z-1|^2 = 2-\sqrt{3} = \dfrac{(\sqrt{3}-1)^2}{2}$ and $|z| = 1$, the distance is $\dfrac{2016}{|z-1|} = 1008\sqrt{6} + 1008\sqrt{2}$.

Hence $a+b+c+d = 1008+6+1008+2 = 2024.$

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc12/2015B