2014 AMC 12B Solutions

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1. Leah has 13 coins, all of which are pennies and nickels. If she had one more nickel than she has now, then she would have the same number of pennies and nickels. In cents, how much are Leah's coins worth?

A 33

B 35

C 37

D 39

E 41

Solution:

Let n be the number of nickels, so Leah has $13 - n$ pennies. One more nickel would give her $n + 1$ nickels, and this equals the number of pennies:

$$n + 1 = 13 - n.$$

Solving gives $n = 6,$ so there are 6 nickels and 7 pennies.

The total value is $6 \cdot 5 + 7 = 37$ cents.

Thus, the correct answer is **C**.

2. Orvin went to the store with just enough money to buy 30 balloons. When he arrived he discovered that the store had a special sale on balloons: buy 1 balloon at the regular price and get a second at $\frac{1}{3}$ off the regular price. What is the greatest number of balloons Orvin could buy?

A 33

B 34

C 36

D 38

E 39

Solution:

Under the sale, a pair of balloons costs $1+\frac{2}{3}=\frac{5}{3}$ times the regular price of one balloon.

Orvin's money buys 30 balloons at the regular price, so he can afford

$$30 \div \frac{5}{3}=18$$

pairs, which is 36 balloons.

Thus, the correct answer is **C**.

3. Randy drove the first third of his trip on a gravel road, the next 20 miles on pavement, and the remaining one-fifth on a dirt road. In miles, how long was Randy's trip?

A 30

B $\frac{400}{11}$

C $\frac{75}{2}$

D 40

E $\frac{300}{7}$

Solution:

The fraction of the trip on pavement is

$$1 - \frac{1}{3} - \frac{1}{5} = \frac{7}{15}.$$

Since this equals 20 miles, the whole trip is

$$20 \div \frac{7}{15} = \frac{300}{7}$$

miles.

Thus, the correct answer is **E**.

4. Susie pays for 4 muffins and 3 bananas. Calvin spends twice as much paying for 2 muffins and 16 bananas. A muffin is how many times as expensive as a banana?

A $\frac{3}{2}$

B $\frac{5}{3}$

C $\frac{7}{4}$

D 2

E $\frac{13}{4}$

Solution:

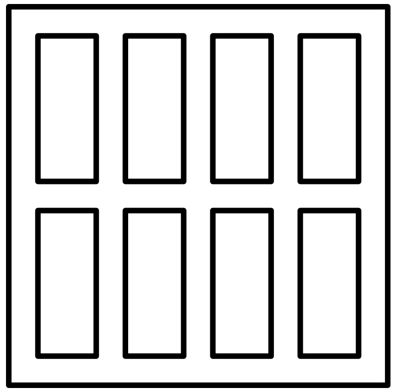
Let a muffin cost m and a banana cost b. Then

$$2(4m + 3b) = 2m + 16b.$$

Expanding gives $8m + 6b = 2m + 16b$, so $6m = 10b$ and $m = \frac{5}{3}b$.

Thus, the correct answer is **B**.

5. Doug constructs a square window using 8 equal-size panes of glass, as shown. The ratio of the height to width for each pane is $5 : 2$, and the borders around and between the panes are 2 inches wide. In inches, what is the side length of the square window?



A 26

B 28

C 30

D 32

E 34

Solution:

Let each pane have width $2x$ and height $5x$. The window is 4 panes wide with 5 vertical borders, so its width is $4(2x) + 5 \cdot 2 = 8x + 10.$

It is 2 panes tall with 3 horizontal borders, so its height is $2(5x) + 3 \cdot 2 = 10x + 6.$

Setting width equal to height gives $8x + 10 = 10x + 6$, so $x = 2$ and the side length is $10 \cdot 2 + 6 = 26.$

Thus, the correct answer is **A**.

6. Ed and Ann both have lemonade with their lunch. Ed orders the regular size. Ann gets the large lemonade, which is 50% more than the regular. After both consume $\frac{3}{4}$ of their drinks, Ann gives Ed a third of what she has left, and 2 additional ounces. When they finish their lemonades they realize that they both drank the same amount. How many ounces of lemonade did they drink together?

- A 30
- B 32
- C 36
- D 40
- E 50

Solution:

Let a regular lemonade hold a ounces, so Ann's large holds $\frac{3}{2}a$. After each drinks $\frac{3}{4}$, Ann has $\frac{1}{4} \cdot \frac{3}{2}a = \frac{3}{8}a$ left, and she gives Ed $\frac{1}{3} \cdot \frac{3}{8}a + 2 = \frac{1}{8}a + 2$ ounces.

Ed drinks his full a ounces plus that gift, and Ann drinks her $\frac{3}{2}a$ minus the gift. Setting these equal,

$$a + \left(\frac{1}{8}a + 2\right) = \frac{3}{2}a - \left(\frac{1}{8}a + 2\right),$$

which gives $4 = \frac{1}{4}a$, so $a = 16.$

Then Ed drank 16 ounces and Ann drank 24 ounces, for a total of 40 ounces.

Thus, the correct answer is **D**.

7. For how many positive integers n is $\dfrac{n}{30-n}$ also a positive integer?

- A 4
- B 5
- C 6
- D 7
- E 8

Solution:

Write

$$\frac{n}{30-n} = \frac{30}{30-n} - 1.$$

For this to be a positive integer, $30-n$ must be a positive divisor of 30 with $\dfrac{30}{30-n} \geq 2$, i.e. $30-n \leq 15.$

The divisors of 30 that are at most 15 are $1, 2, 3, 5, 6, 10, 15$, giving 7 values of n (namely $15, 20, 24, 25, 27, 28, 29$).

Thus, the correct answer is **D**.

8. In the addition shown below $A, B, C,$ and D are distinct digits. How many different values are possible for D?

$$\begin{array}{cccccc} & A & B & B & C & B \\ + & B & C & A & D & A \\ \hline & D & B & D & D & D \end{array}$$

- A 2
- B 4
- C 7
- D 8
- E 9

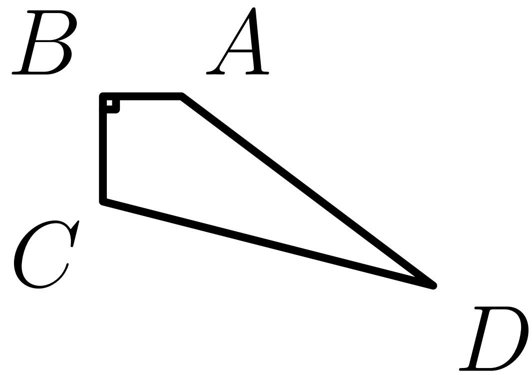
Solution:

The leftmost column shows $A + B = D$ with no carry out, so $A + B \leq 9$. Examining the tens and thousands columns (each of the form $C + \text{digit} + \text{carry}$ producing the same digit) forces $C = 0$ and eliminates all carries.

Every column then reduces to $A + B = D$, with $A, B, C = 0$ distinct. Since A and B are distinct positive digits, $D = A + B$ can be any value from 3 up to 9, giving 7 possibilities, for example $(A, B, C, D) = (1, 2, 0, 3), (1, 3, 0, 4), \ldots, (2, 7, 0, 9)$.

Thus, the correct answer is **C**.

9. Convex quadrilateral $ABCD$ has $AB = 3, BC = 4, CD = 13, AD = 12,$ and $\angle ABC = 90^\circ$, as shown. What is the area of the quadrilateral?



A 30

B 36

C 40

D 48

E 58.5

Solution:

By the Pythagorean Theorem in right triangle ABC, $AC = \sqrt{3^2 + 4^2} = 5.$

Since $5^2 + 12^2 = 13^2$, the converse of the Pythagorean Theorem shows $\angle DAC = 90^\circ$, so $\triangle DAC$ is right.

The area of $\triangle ABC$ is $\frac{1}{2} \cdot 3 \cdot 4 = 6$ and the area of $\triangle DAC$ is $\frac{1}{2} \cdot 5 \cdot 12 = 30.$ The quadrilateral has area $6 + 30 = 36.$

Thus, the correct answer is **B**.

10. Danica drove her new car on a trip for a whole number of hours, averaging 55 miles per hour. At the beginning of the trip, abc miles was displayed on the odometer, where abc is a 3-digit number with $a \geq 1$ and $a+b+c \leq 7$. At the end of the trip, the odometer showed cba miles. What is $a^2+b^2+c^2$?

A 26

B 27

C 36

D 37

E 41

Solution:

The distance driven is $cba - abc = 99(c-a)$, a multiple of 9. Driving a whole number of hours at 55 mph makes it a multiple of 55 too, hence a multiple of 495.

Since the odometer difference is at most a 3-digit number and $a \geq 1$, the distance must be 495, so $c - a = 5$.

With $a \geq 1$ and $a+b+c \leq 7$, the only choice is $a=1, c=6, b=0$. Then $a^2 + b^2 + c^2 = 1 + 0 + 36 = 37$.

Thus, the correct answer is **D**.

11. A list of 11 positive integers has a mean of 10, a median of 9, and a unique mode of 8. What is the largest possible value of an integer in the list?

A 24

B 30

C 31

D 33

E 35

Solution:

The list sums to $11 \cdot 10 = 110$. To maximize one entry, minimize the sum of the other ten.

Sorted, the sixth number must be 9 (the median), and 8 must appear more often than any other value. Trying 8 three times, the smallest possible ten numbers are

$$1, 1, 8, 8, 8, 9, 9, 10, 10, 11,$$

which sum to 75 and keep 8 the unique mode.

The largest entry is then $110 - 75 = 35$.

Thus, the correct answer is **E**.

12. A set S consists of triangles whose sides have integer lengths less than $5,$ and no two elements of S are congruent or similar. What is the largest number of elements that S can have?

A 8

B 9

C 10

D 11

E 12

Solution:

Write each triangle by its side lengths in nonincreasing order. Only one equilateral triangle is allowed (all are similar), and of the similar pair $2, 2, 1$ and $4, 4, 2$ only one may appear.

The remaining valid, pairwise non-similar triangles are

$$443,\ 441,\ 433,\ 432,\ 332,\ 331,\ 322,$$

seven in all. Together with one equilateral and one of the similar pair, S has at most 9 elements.

Thus, the correct answer is **B**.

13. Real numbers a and b are chosen with $1 < a < b$ such that no triangle with positive area has side lengths $1, a,$ and b or $\frac{1}{b}, \frac{1}{a},$ and 1. What is the smallest possible value of b?

A $\dfrac{3+\sqrt{3}}{2}$

B $\dfrac{5}{2}$

C $\dfrac{3+\sqrt{5}}{2}$

D $\dfrac{3+\sqrt{6}}{2}$

E 3

Solution:

Since b is the largest of $1, a, b,$ no such triangle exists exactly when $b \geq a+1$. Since 1 is the largest of $\frac{1}{b}, \frac{1}{a}, 1,$ no such triangle exists exactly when $1 \geq \frac{1}{a} + \frac{1}{b}$, that is $a \leq \frac{b}{b-1}$.

Both conditions hold with b smallest when $a+1=b$ and $a = \frac{b}{b-1}$ meet, giving $b - 1 = \frac{b}{b-1}$, or $b^2 - 3b + 1 = 0$.

The root larger than 1 is $b = \dfrac{3+\sqrt{5}}{2}$.

Thus, the correct answer is **C**.

14. A rectangular box has a total surface area of 94 square inches. The sum of the lengths of all its edges is 48 inches. What is the sum of the lengths in inches of all of its interior diagonals?

A $8\sqrt{3}$

B $10\sqrt{2}$

C $16\sqrt{3}$

D $20\sqrt{2}$

E $40\sqrt{2}$

Solution:

Let the edges be x, y, z. Then $xy + yz + zx = 47$ and $x + y + z = 12$. Therefore

$$x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + yz + zx) = 144 - 94 = 50.$$

Each of the 4 interior diagonals has length $\sqrt{x^2 + y^2 + z^2} = \sqrt{50} = 5\sqrt{2}$, so their total length is $4 \cdot 5\sqrt{2} = 20\sqrt{2}$.

Thus, the correct answer is **D**.

15. When $p = \sum_{k=1}^{6} k \ln k$, the number e^p is an integer. What is the largest power of 2 that is a factor of e^p?

A 2^{12}

B 2^{14}

C 2^{16}

D 2^{18}

E 2^{20}

Solution:

Since $k \ln k = \ln(k^k)$, the sum gives $p = \ln\left(\prod_{k=1}^{6} k^k\right)$, so

$$e^p = 1^1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdot 5^5 \cdot 6^6.$$

The factors of 2 come from 2^2 (giving 2), $4^4 = 2^8$ (giving 8), and $6^6 = 2^6 \cdot 3^6$ (giving 6). In total the exponent of 2 is $2 + 8 + 6 = 16$.

Thus, the correct answer is **C**.

16. Let P be a cubic polynomial with $P(0) = k, P(1) = 2k,$ and $P(-1) = 3k.$ What is $P(2) + P(-2)$?

A 0

B k

C $6k$

D $7k$

E $14k$

Solution:

Since $P(0) = k,$ write $P(x) = ax^3 + bx^2 + cx + k.$

Then $P(1) = a + b + c + k = 2k$ and $P(-1) = -a + b - c + k = 3k.$ Adding these gives $2b + 2k = 5k,$ so $2b = 3k.$

The odd-power terms cancel in the sum:

$$P(2) + P(-2) = (8a + 4b + 2c + k) + (-8a + 4b - 2c + k) = 8b + 2k.$$

Since $8b = 4(2b) = 12k,$ this equals $12k + 2k = 14k.$

Thus, the correct answer is **E**.

17. Let P be the parabola with equation $y = x^2$ and let $Q = (20, 14)$. There are real numbers r and s such that the line through Q with slope m does not intersect P if and only if $r < m < s$. What is $r + s$?

A 1

B 26

C 40

D 52

E 80

Solution:

The line through Q is $y = m(x - 20) + 14$. Substituting into $y = x^2$ gives

$$x^2 - mx + (20m - 14) = 0.$$

There is no intersection exactly when this has no real root, i.e. when the discriminant $m^2 - 4(20m - 14) = m^2 - 80m + 56$ is negative. That happens between the two roots r and s of $m^2 - 80m + 56 = 0$.

By Vieta's formulas, $r + s = 80$.

Thus, the correct answer is **E**.

18. The numbers $1, 2, 3, 4, 5$ are to be arranged in a circle. An arrangement is *bad* if it is not true that for every n from 1 to 15 one can find a subset of the numbers that appear consecutively on the circle that sum to n. Arrangements that differ only by a rotation or a reflection are considered the same. How many different bad arrangements are there?

A 1

B 2

C 3

D 4

E 5

Solution:

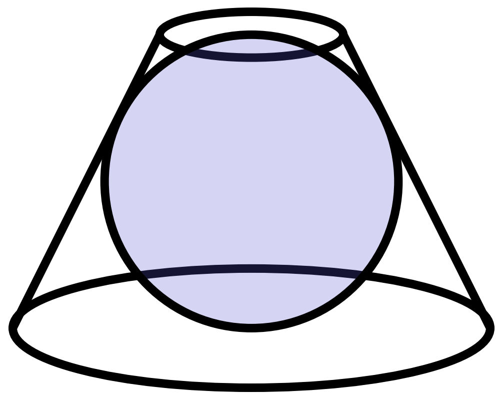
Any single number covers sums 1 through 5. If a consecutive block sums to n, the remaining numbers form a consecutive block summing to $15 - n$, so sums 10 through 14 are automatically covered as well. Thus an arrangement is bad only if it fails to produce 6 or 7.

If 6 cannot be formed, then 1 and 5 are not adjacent, and working through the cases forces the arrangement 14352. If 7 cannot be formed, then 2 and 5 are not adjacent, forcing 23154.

These are the only two bad arrangements up to rotation and reflection.

Thus, the correct answer is **B**.

19. A sphere is inscribed in a truncated right circular cone as shown. The volume of the truncated cone is twice that of the sphere. What is the ratio of the radius of the bottom base of the truncated cone to the radius of the top base of the truncated cone?



A $\frac{3}{2}$

B $\frac{1+\sqrt{5}}{2}$

C $\sqrt{3}$

D 2

E $\frac{3+\sqrt{5}}{2}$

Solution:

Let the top radius be 1, the bottom radius r, and the sphere radius a. The sphere touches both bases, so the cone's height is $2a$, and applying the Pythagorean Theorem to the side profile gives $r = a^2$.

The frustum volume is $\frac{1}{3}\pi(r^2 + r + 1)(2a)$. Setting it equal to twice the sphere volume $\frac{4}{3}\pi a^3$ and using $r = a^2$ yields

$$a^4 - 3a^2 + 1 = 0,$$

that is $r^2 - 3r + 1 = 0$.

The positive root is $r = \dfrac{3+\sqrt{5}}{2}$.

Thus, the correct answer is **E**.

20. For how many positive integers x is $\log_{10}(x - 40) + \log_{10}(60 - x) < 2$?

A 10

B 18

C 19

D 20

E infinitely many

Solution:

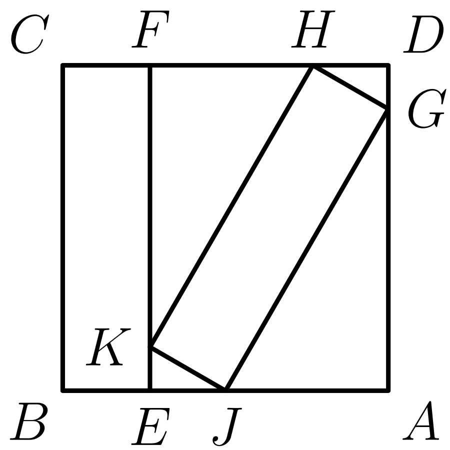
The logarithms are defined only when $x - 40 > 0$ and $60 - x > 0$, so $40 < x < 60$.

Within this range the inequality becomes $(x - 40)(60 - x) < 100$, which expands to $x^2 - 100x + 2500 > 0$, i.e. $(x - 50)^2 > 0$. This holds for every $x \neq 50$.

The integers strictly between 40 and 60 except 50 are $41, \ldots, 49$ and $51, \ldots, 59$, which is 18 values.

Thus, the correct answer is **B**.

21. In the figure, $ABCD$ is a square of side length $1.$ The rectangles $JKHG$ and $EBCF$ are congruent. What is BE?



A $\frac{1}{2}(\sqrt{6}-2)$

B $\frac{1}{4}$

C $2-\sqrt{3}$

D $\frac{\sqrt{3}}{6}$

E $1-\frac{\sqrt{2}}{2}$

Solution:

Let $x = BE = GH = CF$ and $\theta = \angle DHG = \angle AGJ = \angle FKH,$ with $AD = GJ = HK = 1.$ In right triangle $GDH, x\sin\theta = DG = 1-\cos\theta,$ so $x = \dfrac{1-\cos\theta}{\sin\theta}.$

Along side CD,

$$1 = CF + FH + HD = x + \sin\theta + x\cos\theta.$$

Substituting for x gives

$$1 = \frac{(1-\cos\theta)(1+\cos\theta)}{\sin\theta} + \sin\theta = \frac{\sin^2\theta}{\sin\theta} + \sin\theta = 2\sin\theta.$$

Hence $\sin\theta = \frac{1}{2}$, so $\theta = 30^\circ$ and

$$x = \frac{1 - \frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2 - \sqrt{3}.$$

Thus, the correct answer is **C**.

22. In a small pond there are eleven lily pads in a row labeled 0 through 10. A frog is sitting on pad 1. When the frog is on pad $N, 0 < N < 10$, it will jump to pad $N-1$ with probability $\dfrac{N}{10}$ and to pad $N+1$ with probability $1-\dfrac{N}{10}$. Each jump is independent of the previous jumps. If the frog reaches pad 0 it will be eaten by a patiently waiting snake. If the frog reaches pad 10 it will exit the pond, never to return. What is the probability that the frog will escape being eaten by the snake?

A $\dfrac{32}{79}$

B $\dfrac{161}{384}$

C $\dfrac{63}{146}$

D $\dfrac{7}{16}$

E $\dfrac{1}{2}$

Solution:

Let p_j be the probability of eventually reaching pad 10 starting from pad j. By the symmetry of the jump rule at the center, $p_5 = \frac{1}{2}$.

Each interior pad satisfies $p_j = \frac{10-j}{10}\, p_{j+1} + \frac{j}{10}\, p_{j-1}$, which gives

$$p_4 = \tfrac{2}{5}p_3 + \tfrac{3}{5}p_5, \quad p_3 = \tfrac{3}{10}p_2 + \tfrac{7}{10}p_4,$$

$$p_2 = \tfrac{1}{5}p_1 + \tfrac{4}{5}p_3, \quad p_1 = \tfrac{9}{10}p_2.$$

Substituting downward from $p_5 = \frac{1}{2}$ and solving yields $p_1 = \dfrac{63}{146}$.

Thus, the correct answer is **C**.

23. The number 2017 is prime. Let $S=\sum_{k=0}^{62}\binom{2014}{k}$. What is the remainder when S is divided by 2017?

A 32

B 684

C 1024

D 1576

E 2016

Solution:

Working modulo 2017, the identity $n\cdot k!\cdot(2014-k)!=2014!$ together with $2016\cdot 2015\cdots(2015-k)\equiv(-1)^k(k+2)!$ leads to

$$2\binom{2014}{k}\equiv(-1)^k(k+2)(k+1)\pmod{2017},$$

so $\binom{2014}{k}\equiv(-1)^k\binom{k+2}{2}$.

Then

$$S\equiv\sum_{k=0}^{62}(-1)^k\binom{k+2}{2}=1+\sum_{k=1}^{31}\left[\binom{2k+2}{2}-\binom{2k+1}{2}\right]=1+\sum_{k=1}^{31}(2k+1).$$

The remainding sum is $3+5+\cdots+63=1023$, so $S\equiv 1+1023=1024$ $\pmod{2017}$.

Thus, the correct answer is **C**.

24. Let $ABCDE$ be a pentagon inscribed in a circle such that $AB = CD = 3, BC = DE = 10$, and $AE = 14.$ The sum of the lengths of all diagonals of $ABCDE$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

A 129

B 247

C 353

D 391

E 421

Solution:

Because arcs AB, CD are equal and arcs BC, DE are equal, the chords AC, BD, CE are all equal; let $x = AC = BD = CE, y = AD$, and $z = BE.$

Ptolemy's theorem on $ABCD, BCDE$, and $ABDE$ gives

$$10y + 9 = x^2, \quad 100 + 3z = x^2, \quad 30 + 14x = yz.$$

Solving the first two for y and z and substituting into the third yields

$$x^3 - 109x - 420 = 0 = (x - 12)(x + 5)(x + 7).$$

So $x = 12, y = \frac{135}{10} = \frac{27}{2}$, and $z = \frac{44}{3}$. The five diagonals are AC, BD, CE, AD, BE, summing to

$$3x + y + z = 36 + \frac{27}{2} + \frac{44}{3} = \frac{385}{6}.$$

Thus $m + n = 385 + 6 = 391$, and the correct answer is **D**.

25. What is the sum of all positive real solutions x to the equation

$$2\cos(2x)\left(\cos(2x)-\cos\left(\frac{2014\pi^2}{x}\right)\right)=\cos(4x)-1?$$

A π

B 810π

C 1008π

D 1080π

E 1800π

Solution:

Let $x=\frac{\pi y}{2}$. Dividing by 2 and using $\frac{1}{2}(1-\cos(2\pi y))=\sin^2(\pi y)$, the equation simplifies to

$$\cos(\pi y)\cos\left(\frac{4028\pi}{y}\right)=1.$$

Both cosines must equal 1 or both equal -1, so y and $\frac{4028}{y}$ are integers of the same parity. Since $4028=2^2\cdot 19\cdot 53$ is even, both must be even, so $y=2a$ with a a positive odd divisor of $2014=2\cdot 19\cdot 53$, giving $a\in\{1,19,53,19\cdot 53\}$.

Each such a gives $x=\frac{\pi y}{2}=\pi a$, so the sum of solutions is

$$\pi(1+19+53+19\cdot 53)=\pi(19+1)(53+1)=1080\pi.$$

Thus, the correct answer is **D**.

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