2014 AMC 12A Solutions

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1. What is

$$10 \cdot \left(\frac{1}{2} + \frac{1}{5} + \frac{1}{10}\right)^{-1} ?$$

A 3

B 8

C $\frac{25}{2}$

D $\frac{170}{3}$

E 170

Solution:

The sum inside the parentheses is $\frac{1}{2} + \frac{1}{5} + \frac{1}{10} = \frac{5+2+1}{10} = \frac{4}{5}$.

Its reciprocal is $\frac{5}{4}$, so the expression equals $10 \cdot \frac{5}{4} = \frac{25}{2}$.

Thus, the correct answer is **C**.

2. At the theater children get in for half price. The price for 5 adult tickets and 4 child tickets is $\$24.50$. How much would 8 adult tickets and 6 child tickets cost?

A $\$35$

B $\$38.50$

C $\$40$

D $\$42$

E $\$42.50$

Solution:

Since a child ticket is half an adult ticket, 5 adult and 4 child tickets equal $5 + \frac{1}{2} \cdot 4 = 7$ adult tickets, so one adult ticket costs $\dfrac{24.50}{7} = \$3.50.$

The second purchase equals $8 + \frac{1}{2} \cdot 6 = 11$ adult tickets, costing $11 \cdot 3.50 = \$38.50.$

Thus, the correct answer is **B**.

3. Walking down Jane Street, Ralph passed four houses in a row, each painted a different color. He passed the orange house before the red house, and he passed the blue house before the yellow house. The blue house was not next to the yellow house. How many orderings of the colored houses are possible?

- A 2
- B 3
- C 4
- D 5
- E 6

Solution:

If orange comes first, then blue and yellow cannot be adjacent, forcing the order orange, blue, red, yellow.

If blue comes first, yellow can be in the third or fourth position (never second, to avoid adjacency), giving blue, orange, yellow, red and blue, orange, red, yellow.

These are the only 3 valid orderings.

Thus, the correct answer is **B**.

4. Suppose that a cows give b gallons of milk in c days. At this rate, how many gallons of milk will d cows give in e days?

A $\frac{bde}{ac}$

B $\frac{ac}{bde}$

C $\frac{abde}{c}$

D $\frac{bcde}{a}$

E $\frac{abc}{de}$

Solution:

The rate is $\frac{b}{ac}$ gallons per cow per day.

So d cows over e days produce $\frac{b}{ac} \cdot d \cdot e = \frac{bde}{ac}$ gallons.

Thus, the correct answer is **A**.

5. On an algebra quiz, 10% of the students scored 70 points, 35% scored 80 points, 30% scored 90 points, and the rest scored 100 points. What is the difference between the mean and the median of the students' scores on this quiz?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

The remaining 25% scored $100.$ Since 45% scored at most 80 and 75% scored at most $90,$ the median is $90.$

The mean is $0.10(70) + 0.35(80) + 0.30(90) + 0.25(100) = 7 + 28 + 27 + 25 = 87.$

The difference is $90 - 87 = 3.$

Thus, the correct answer is **C**.

6. The difference between a two-digit number and the number obtained by reversing its digits is 5 times the sum of the digits of either number. What is the sum of the two-digit number and its reverse?

- A 44
- B 55
- C 77
- D 99
- E 110

Solution:

Let the larger number be $10a + b$. Then

$$(10a + b) - (10b + a) = 9(a - b) = 5(a + b),$$

which simplifies to $2a = 7b$.

The only nonzero digits satisfying this are $a = 7$ and $b = 2$, so the number is 72 and its reverse is 27.

Their sum is $72 + 27 = 99$.

Thus, the correct answer is **D**.

7. The first three terms of a geometric progression are $\sqrt{3}$, $\sqrt[3]{3}$, and $\sqrt[6]{3}$. What is the fourth term?

A 1

B $\sqrt[7]{3}$

C $\sqrt[8]{3}$

D $\sqrt[9]{3}$

E $\sqrt[10]{3}$

Solution:

Writing the terms as powers of 3, they are $3^{1/2}, 3^{1/3}, 3^{1/6}$. The common ratio is $\frac{3^{1/3}}{3^{1/2}} = 3^{-1/6}$.

The fourth term is $3^{1/6} \cdot 3^{-1/6} = 3^0 = 1$.

Thus, the correct answer is **A**.

8. A customer who intends to purchase an appliance has three coupons, only one of which may be used:

Coupon 1: 10% off the listed price if the listed price is at least $\$50$

Coupon 2: $\$20$ off the listed price if the listed price is at least $\$100$

Coupon 3: 18% off the amount by which the listed price exceeds $\$100$

For which of the following listed prices will coupon 1 offer a greater price reduction than either coupon 2 or coupon 3?

- A $\$179.95$
- B $\$199.95$
- C $\$219.95$
- D $\$239.95$
- E $\$259.95$

Solution:

For a price $P > 100$, the reductions are $\frac{P}{10}$, 20, and $\frac{18}{100}(P - 100)$.

Coupon 1 beats coupon 2 when $\frac{P}{10} > 20$, that is $P > 200$. Coupon 1 beats coupon 3 when $\frac{P}{10} > \frac{18}{100}(P - 100)$, that is $P < 225$.

The only listed price in $(200, 225)$ is $\$219.95$.

Thus, the correct answer is **C**.

9. Five positive consecutive integers starting with a have average b. What is the average of 5 consecutive integers that start with b?

- A $a + 3$
- B $a + 4$
- C $a + 5$
- D $a + 6$
- E $a + 7$

Solution:

The integers $a, a + 1, a + 2, a + 3, a + 4$ have average $a + 2,$ so $b = a + 2.$

The integers starting at b have average $b + 2 = (a + 2) + 2 = a + 4.$

Thus, the correct answer is **B**.

10. Three congruent isosceles triangles are constructed with their bases on the sides of an equilateral triangle of side length 1. The sum of the areas of the three isosceles triangles is the same as the area of the equilateral triangle. What is the length of one of the two congruent sides of one of the isosceles triangles?

A $\frac{\sqrt{3}}{4}$

B $\frac{\sqrt{3}}{3}$

C $\frac{2}{3}$

D $\frac{\sqrt{2}}{2}$

E $\frac{\sqrt{3}}{2}$

Solution:

The equilateral triangle has area $\frac{\sqrt{3}}{4}$. Each isosceles triangle has base 1 and height h, so $3 \cdot \frac{1}{2}h = \frac{\sqrt{3}}{4}$, giving $h = \frac{\sqrt{3}}{6}$.

A congruent side is the hypotenuse from the apex to a base endpoint:

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{12}} = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}.$$

Thus, the correct answer is **B**.

11. David drives from his home to the airport to catch a flight. He drives 35 miles in the first hour, but realizes that he will be 1 hour late if he continues at this speed. He increases his speed by 15 miles per hour for the rest of the way to the airport and arrives 30 minutes early. How many miles is the airport from his home?

- A 140
- B 175
- C 210
- D 245
- E 280

Solution:

Let d be the remaining distance after one hour and t the remaining time until the flight. At 35 mph he would be an hour late, so $d = 35(t+1)$. At 50 mph he is half an hour early, so $d = 50\left(t - \frac{1}{2}\right)$.

Setting these equal gives $35t + 35 = 50t - 25$, so $t = 4$ and $d = 175$.

The total distance is $175 + 35 = 210$ miles.

Thus, the correct answer is **C**.

12. Two circles intersect at points A and B. The minor arcs AB measure 30° on one circle and 60° on the other circle. What is the ratio of the area of the larger circle to the area of the smaller circle?

- A 2
- B $1+\sqrt{3}$
- C 3
- D $2+\sqrt{3}$
- E 4

Solution:

Let the circles have radii R (with the 30° arc) and r (with the 60° arc). The common chord has length $2R\sin 15^\circ = 2r\sin 30^\circ$, so $\dfrac{R}{r} = \dfrac{\sin 30^\circ}{\sin 15^\circ}$.

The smaller central angle gives the larger radius, so $R > r$. The area ratio is

$$\left(\frac{R}{r}\right)^2 = \frac{1}{4\sin^2 15^\circ} = \frac{1}{2(1-\cos 30^\circ)} = \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}.$$

Thus, the correct answer is **D**.

13. A fancy bed and breakfast inn has 5 rooms, each with a distinctive color-coded decor. One day 5 friends arrive to spend the night. There are no other guests that night. The friends can room in any combination they wish, but with no more than 2 friends per room. In how many ways can the innkeeper assign the guests to the rooms?

- A 2100
- B 2220
- C 3000
- D 3120
- E 3125

Solution:

All singles: assign 5 friends to 5 rooms in $5! = 120$ ways.

One pair: choose the pair in $\binom{5}{2} = 10$ ways, then place the 4 groups into rooms in $5 \cdot 4 \cdot 3 \cdot 2 = 120$ ways, giving $10 \cdot 120 = 1200$.

Two pairs: choose the solo friend in 5 ways and split the rest into two pairs in 3 ways (15 groupings), then place the 3 groups into rooms in $5 \cdot 4 \cdot 3 = 60$ ways, giving $15 \cdot 60 = 900$.

The total is $120 + 1200 + 900 = 2220$.

Thus, the correct answer is **B**.

14. Let $a < b < c$ be three integers such that a, b, c is an arithmetic progression and a, c, b is a geometric progression. What is the smallest possible value for c?

- A -2
- B 1
- C 2
- D 4
- E 6

Solution:

Let $d = b - a > 0$, so $b = a + d$ and $c = a + 2d$. Since a, c, b is geometric,

$$\frac{c}{a} = \frac{b}{c} \implies (a + 2d)^2 = a(a + d),$$

which simplifies to $3ad + 4d^2 = 0$, so $3a + 4d = 0$.

Then $a = -4k$ and $d = 3k$ for a positive integer k, giving $c = a + 2d = 2k$. The smallest value is $c = 2$ (with $a = -4, b = -1, c = 2$).

Thus, the correct answer is **C**.

15. A five-digit palindrome is a positive integer with respective digits $abcba$, where a is not zero. Let S be the sum of all five-digit palindromes. What is the sum of the digits of S?

- A 9
- **B** 18
- C 27
- D 36
- E 45

Solution:

Write $\overline{abcba} = 10001a + 1010b + 100c$. Summing over all palindromes, each value of $a \in \{1, \ldots, 9\}$ occurs with $10 \cdot 10 = 100$ choices of b, c, and each value of b or c occurs with $9 \cdot 10 = 90$ choices of the other two digits.

Using $\sum a = \sum b = \sum c = 45$,

$$S = 45\big(10001 \cdot 100 + 1010 \cdot 90 + 100 \cdot 90\big) = 45 \cdot 1{,}100{,}000 = 49{,}500{,}000.$$

The sum of the digits of S is $4 + 9 + 5 = 18$.

Thus, the correct answer is **B**.

16. The product $(8)(888\ldots8)$, where the second factor has k digits, is an integer whose digits have a sum of 1000. What is k?

A 901

B 911

C 919

D 991

E 999

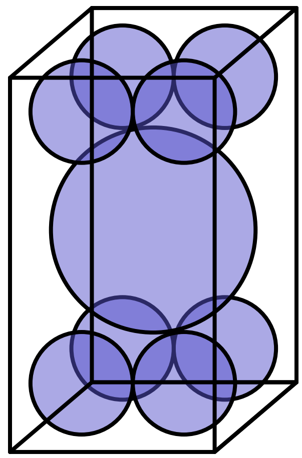
Solution:

By carrying out the multiplication, $8 \cdot \underbrace{88\ldots8}_{k} = 7\underbrace{1\ldots1}_{k-2}04$, which has $k-2$ ones.

The digit sum is $7 + (k-2) + 0 + 4 = k + 9$. Setting $k + 9 = 1000$ gives $k = 991.$

Thus, the correct answer is **D**.

17. A $4 \times 4 \times h$ rectangular box contains a sphere of radius 2 and eight smaller spheres of radius 1. The smaller spheres are each tangent to three sides of the box, and the larger sphere is tangent to each of the smaller spheres. What is h?



A $2+2\sqrt{7}$

B $3+2\sqrt{5}$

C $4+2\sqrt{7}$

D $4\sqrt{5}$

E $4\sqrt{7}$

Solution:

Place the box with a corner at the origin. Each small sphere sits in a corner with center 1 unit from three faces. The four top small-sphere centers form a square of side 2, whose center lies on the box axis; a corner of that square is $\sqrt{2}$ from the center.

The big sphere's center is on the axis, at distance $2+1=3$ from each top small center. The vertical gap between them is $\sqrt{3^2-\sqrt{2}^2}=\sqrt{7}$.

The big center is at height $\frac{h}{2}$ and the top small centers at height $h-1$, so $(h-1)-\frac{h}{2}=\sqrt{7}$, giving $\frac{h}{2}=1+\sqrt{7}$ and $h=2+2\sqrt{7}$.

Thus, the correct answer is **A**.

18. The domain of the function

$$f(x) = \log_{1/2}\left(\log_4\left(\log_{1/4}\left(\log_{16}\left(\log_{1/16} x\right)\right)\right)\right)$$

is an interval of length $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

A 19

B 31

C 271

D 319

E 511

Solution:

Working from the outside, f is defined exactly when $\log_4\left(\log_{1/4}\left(\log_{16}\left(\log_{1/16} x\right)\right)\right) > 0$, which is equivalent to $\log_{1/4}\left(\log_{16}\left(\log_{1/16} x\right)\right) > 1.$

Since the base $\frac{1}{4} < 1$, this means $0 < \log_{16}\left(\log_{1/16} x\right) < \frac{1}{4}$, hence $1 < \log_{1/16} x < 16^{1/4} = 2.$

As $\frac{1}{16} < 1$, this reverses to $\left(\frac{1}{16}\right)^2 < x < \left(\frac{1}{16}\right)^1$, i.e. $\frac{1}{256} < x < \frac{1}{16}$. The length is $\frac{1}{16} - \frac{1}{256} = \frac{15}{256}$, so $m + n = 15 + 256 = 271.$

Thus, the correct answer is **C**.

19. There are exactly N distinct rational numbers k such that $|k| < 200$ and

$$5x^2 + kx + 12 = 0$$

has at least one integer solution for x. What is N?

- A 6
- B 12
- C 24
- D 48
- E 78

Solution:

If an integer x is a root, then $k = -\left(5x + \frac{12}{x}\right)$, so $x \neq 0$. For $x \geq 2$, $|k| = 5|x| + \frac{12}{|x|}$ increases, and $|x| = 39$ gives $|k| \approx 195.3 < 200$, while $|x| = 40$ gives $|k| > 200$.

Thus x ranges over $\pm 1, \pm 2, \ldots, \pm 39$, which is 78 values. If two integers $a \neq b$ gave the same k, then $5a + \frac{12}{a} = 5b + \frac{12}{b}$ forces $5ab = 12$, which has no integer solutions, so all 78 values of k are distinct.

Thus, the correct answer is **E**.

20. In $\triangle BAC, \angle BAC = 40^\circ, AB = 10$, and $AC = 6$. Points D and E lie on $\overline{AB}$ and $\overline{AC}$, respectively. What is the minimum possible value of $BE + DE + CD$?

A $6\sqrt{3} + 3$

B $\frac{27}{2}$

C $8\sqrt{3}$

D 14

E $3\sqrt{3} + 9$

Solution:

Reflect B across line AC to get B', and reflect C across line AB to get C'. Then $BE = B'E$ and $CD = C'D$, so $BE + DE + CD = B'E + ED + DC'$, a broken path from B' to C'.

This is minimized when the path is the straight segment $B'C'$. We have $AB' = AB = 10, AC' = AC = 6$, and $\angle B'AC' = 3 \cdot 40^\circ = 120^\circ$.

By the Law of Cosines,

$$B'C'^2 = 10^2 + 6^2 - 2 \cdot 10 \cdot 6 \cos 120^\circ = 136 + 60 = 196,$$

so $B'C' = 14$.

Thus, the correct answer is **D**.

21. For every real number x, let $\lfloor x \rfloor$ denote the greatest integer not exceeding x, and let

$$f(x) = \lfloor x \rfloor \left(2014^{x - \lfloor x \rfloor} - 1\right).$$

The set of all numbers x such that $1 \leq x < 2014$ and $f(x) \leq 1$ is a union of disjoint intervals. What is the sum of the lengths of those intervals?

A 1

B $\dfrac{\log 2015}{\log 2014}$

C $\dfrac{\log 2014}{\log 2013}$

D $\dfrac{2014}{2013}$

E $2014^{1/2014}$

Solution:

Write $x = n + r$ with integer n ($1 \leq n \leq 2013$) and $0 \leq r < 1$. Then $f(x) = n\left(2014^r - 1\right)$, and $f(x) \leq 1$ becomes $2014^r \leq 1 + \dfrac{1}{n}$, i.e. $0 \leq r \leq \log_{2014} \dfrac{n+1}{n}$.

Each n contributes an interval of length $\log_{2014} \dfrac{n+1}{n}$, so the total is

$$\sum_{n=1}^{2013} \log_{2014} \frac{n+1}{n} = \log_{2014}\left(\frac{2}{1} \cdot \frac{3}{2} \cdots \frac{2014}{2013}\right) = \log_{2014} 2014 = 1.$$

Thus, the correct answer is **A**.

22. The number 5^{867} is between 2^{2013} and 2^{2014}. How many pairs of integers (m, n) are there such that $1 \leq m \leq 2012$ and

$$5^n < 2^m < 2^{m+2} < 5^{n+1}?$$

A 278

B 279

C 280

D 281

E 282

Solution:

Because $2^2 < 5 < 2^3$, each interval $(5^n, 5^{n+1})$ contains either two or three powers of 2. The chain $5^n < 2^m < 2^{m+2} < 5^{n+1}$ holds exactly when the interval contains three consecutive powers of 2, and then there is a unique such m.

Let d and t be the numbers of intervals $(5^n, 5^{n+1})$ for $0 \leq n \leq 866$ containing two and three powers of 2, respectively. Since $2^{2013} < 5^{867} < 2^{2014}$ there are 2013 powers of 2 in total, giving $d + t = 867$ and $2d + 3t = 2013$.

Solving, $t = 2013 - 2 \cdot 867 = 279$.

Thus, the correct answer is **B**.

23. The fraction

$$\frac{1}{99^2} = 0.\overline{b_{n-1}b_{n-2}\ldots b_2b_1b_0},$$

where n is the length of the period of the repeating decimal expansion. What is the sum $b_0 + b_1 + \cdots + b_{n-1}$?

- A 874
- B 883
- C 887
- D 891
- E 892

Solution:

Reading the block in pairs of digits (base 100), $\frac{1}{9801} = \frac{1}{99^2}$ expands as $00, 01, 02, \ldots,$ since $\frac{1}{(100-1)^2} = \sum_{k\geq 1} k \cdot 100^{-k}$. Carrying works out so that every two-digit block $00, 01, \ldots, 97$ appears, the block 98 is skipped, and 99 appears, before the period repeats.

If the blocks 00 through 99 all appeared, the digit sum would be $(0+1+\cdots+9)\cdot 20 = 900$. Removing the missing 98 subtracts $9+8$, giving $900-9-8=883.$

Thus, the correct answer is **B**.

24. Let $f_0(x) = x + |x - 100| - |x + 100|$, and for $n \geq 1$, let $f_n(x) = |f_{n-1}(x)| - 1$. For how many values of x is $f_{100}(x) = 0$?

A 299

B 300

C 301

D 302

E 303

Solution:

If $f_{n-1}(x) = \pm k$, then $f_n(x) = k - 1$. So if $f_0(x) = \pm k$ for a nonnegative integer k, then $f_k(x) = 0$, after which the sequence alternates $0, -1, 0, \ldots$ Thus $f_{100}(x) = 0$ exactly when $f_0(x) = 2k$ for some integer $-50 \leq k \leq 50$.

Now $f_0(x) = x + |x - 100| - |x + 100|$ equals $x + 200$ for $x < -100$, $-x$ for $-100 \leq x < 100$, and $x - 200$ for $x \geq 100$. Its graph is piecewise linear with turning points $(-100, 100)$ and $(100, -100)$.

A line $y = 2k$ meets this graph three times for $-49 \leq k \leq 49$ and twice for $k = \pm 50$. The total is $99 \cdot 3 + 2 \cdot 2 = 301$.

Thus, the correct answer is **C**.

25. The parabola P has focus $(0,0)$ and goes through the points $(4,3)$ and $(-4,-3)$. For how many points $(x,y) \in P$ with integer coordinates is it true that $|4x+3y| \leq 1000$?

A 38

B 40

C 42

D 44

E 46

Solution:

Since $(0,0)$ is the midpoint of $A=(4,3)$ and $B=(-4,-3)$, the segment AB is the latus rectum, so the directrix is parallel to AB at distance 5 on the far side, namely $4y-3x+25=0$.

Equating distances to focus and directrix gives $(4x+3y)^2 = 25\big(25+2(4y-3x)\big)$. Writing $4x+3y=5s$ forces $5 \mid s$, and $s=5t$ forces t odd; with $t=2u+1$ the integer points are

$$x=-6u^2+2u+4, \qquad y=8u^2+14u+3.$$

Then $|4x+3y| = |50u+25| \leq 1000$ iff $|2u+1| \leq 39$, i.e. $-20 \leq u \leq 19$. That gives 40 lattice points.

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc12/2014A