2013 AMC 12B Solutions

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1. On a particular January day, the high temperature in Lincoln, Nebraska, was 16 degrees higher than the low temperature, and the average of the high and low temperatures was 3°. In degrees, what was the low temperature in Lincoln that day?

A -13

B -8

C -5

D -3

E 11

Solution:

The high exceeds the low by 16, so the low is 8 below the average. Since the average is 3°, the low temperature is $3 - 8 = -5^\circ$. Thus, the correct answer is **C**.

2. Mr. Green measures his rectangular garden by walking two of the sides and finds that it is 15 steps by 20 steps. Each of Mr. Green's steps is 2 feet long. Mr. Green expects a half a pound of potatoes per square foot from his garden. How many pounds of potatoes does Mr. Green expect from his garden?

- A 600
- B 800
- C 1000
- D 1200
- E 1400

Solution:

The garden is $2 \cdot 15 = 30$ feet by $2 \cdot 20 = 40$ feet, an area of 1200 square feet. At half a pound per square foot, Mr. Green expects $\frac{1}{2} \cdot 1200 = 600$ pounds. Thus, the correct answer is **A**.

3. When counting from 3 to 201, 53 is the 51st number counted. When counting backwards from 201 to 3, 53 is the nth number counted. What is n?

A 146

B 147

C 148

D 149

E 150

Solution:

Counting down from 201, the value x is the $(202-x)$th number. So 53 is the $(202-53)=149$th number. Thus, the correct answer is **D**.

4. Ray's car averages 40 miles per gallon of gasoline, and Tom's car averages 10 miles per gallon of gasoline. Ray and Tom each drive the same number of miles. What is the cars' combined rate of miles per gallon of gasoline?

A 10

B 16

C 25

D 30

E 40

Solution:

If each drives D miles, together they cover $2D$ miles using $\dfrac{D}{40}+\dfrac{D}{10}=\dfrac{D}{8}$ gallons. The combined rate is $\dfrac{2D}{D/8}=16$ miles per gallon. Thus, the correct answer is **B**.

5. The average age of 33 fifth-graders is 11. The average age of 55 of their parents is 33. What is the average age of all of these parents and fifth-graders?

- A 22
- B 23.25
- C 24.75
- D 26.25
- E 28

Solution:

The parents' ages sum to $55 \cdot 33$ and the fifth-graders' to $33 \cdot 11$, a total of $33 \cdot 66$. Dividing by 88 people gives $\frac{33 \cdot 66}{88} = 24.75$. Thus, the correct answer is **C**.

6. Real numbers x and y satisfy the equation $x^2 + y^2 = 10x - 6y - 34$. What is $x + y$?

- A 1
- B 2
- C 3
- D 6
- E 8

Solution:

Rearranging gives $x^2 - 10x + 25 + y^2 + 6y + 9 = 0$, that is $(x-5)^2 + (y+3)^2 = 0$. Hence $x = 5$ and $y = -3$, so $x + y = 2$. Thus, the correct answer is **B**.

7. Jo and Blair take turns counting from 1 to one more than the last number said by the other person. Jo starts by saying "1", so Blair follows by saying "1, 2". Jo then says "1, 2, 3", and so on. What is the 53rd number said?

- A 2
- B 3
- C 5
- D 6
- E 8

Solution:

After the turn that counts up to n, exactly $1+2+\cdots+n=\frac{1}{2}n(n+1)$ numbers have been said. For $n=9$ that is $45.$ The next turn starts $1, 2, \ldots,$ so the 53rd number is the 8th number of that turn, namely $8.$ Thus, the correct answer is **E**.

8. Line ℓ_1 has equation $3x - 2y = 1$ and goes through $A = (-1, -2)$. Line ℓ_2 has equation $y = 1$ and meets line ℓ_1 at point B. Line ℓ_3 has positive slope, goes through point A, and meets ℓ_2 at point C. The area of $\triangle ABC$ is 3. What is the slope of ℓ_3?

A $\frac{2}{3}$

B $\frac{3}{4}$

C 1

D $\frac{4}{3}$

E $\frac{3}{2}$

Solution:

Solving $3x - 2y = 1$ with $y = 1$ gives $B = (1, 1)$. The distance from $A = (-1, -2)$ to the line $y = 1$ is 3, so $\frac{1}{2} \cdot BC \cdot 3 = 3$ gives $BC = 2$. Then $C = (3, 1)$ or $C = (-1, 1)$; the latter makes ℓ_3 vertical, so $C = (3, 1)$ and the slope is $\dfrac{1 - (-2)}{3 - (-1)} = \dfrac{3}{4}$.

Thus, the correct answer is **B**.

9. What is the sum of the exponents of the prime factors of the square root of the largest perfect square that divides $12!$?

A 5

B 7

C 8

D 10

E 12

Solution:

Since $12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11,$ the largest perfect square dividing it is $2^{10} \cdot 3^4 \cdot 5^2,$ whose square root is $2^5 \cdot 3^2 \cdot 5.$ The exponents sum to $5 + 2 + 1 = 8.$ Thus, the correct answer is **C**.

10. Alex has 75 red tokens and 75 blue tokens. There is a booth where Alex can give two red tokens and receive in return a silver token and a blue token, and another booth where Alex can give three blue tokens and receive in return a silver token and a red token. Alex continues to exchange tokens until no more exchanges are possible. How many silver tokens will Alex have at the end?

- A 62
- B 82
- C 83
- D 102
- E 103

Solution:

After m red-booth and n blue-booth exchanges, Alex has $75-(2m-n)$ red tokens, $75-(3n-m)$ blue tokens, and $m+n$ silver tokens. Exchanges are impossible exactly when $2m-n \geq 74$ and $3n-m \geq 73$. Equality holds at $(m,n)=(59,44)$, giving $59+44=103$ silver tokens. Thus, the correct answer is **E**.

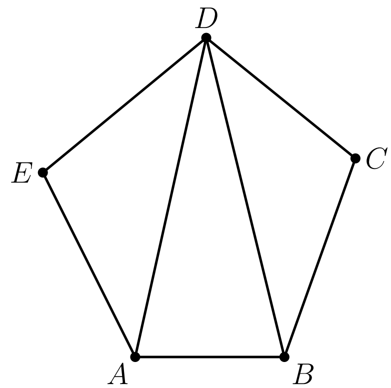
11. Two bees start at the same spot and fly at the same rate in the following directions. Bee A travels 1 foot north, then 1 foot east, then 1 foot upwards, and then continues to repeat this pattern. Bee B travels 1 foot south, then 1 foot west, and then continues to repeat this pattern. In what directions are the bees traveling when they are exactly 10 feet away from each other?

- A A east, B west
- B A north, B south
- C A north, B west
- D A up, B south
- E A up, B west

Solution:

Take east, north, up as x, y, z. After 7 feet bee A is at $(2, 3, 2)$ and bee B is at $(-3, -4, 0)$, a distance $\sqrt{78} < 10$. On the next foot bee A moves east to $(3, 3, 2)$ and bee B moves west to $(-4, -4, 0)$, a distance $\sqrt{102} > 10$. So they pass through 10 feet apart while A heads east and B heads west. Thus, the correct answer is **A**.

12. Cities A, B, C, D, and E are connected by roads AB, AD, AE, BC, BD, CD, and DE. How many different routes are there from A to B that use each road exactly once? (Such a route will necessarily visit some cities more than once.)



A 7

B 9

C 12

D 16

E 18

Solution:

City E (roads AE, DE) is a detour on an A–D trip, and city C (roads BC, CD) is a detour on a B–D trip. Replace them to get a graph on A, B, D with two A–D connections, two B–D connections, and one A–B road. The trails from A to B using each once are of 4 types: $ABDADB$, $ADABDB$, $ADBADB$, and $ADBDAB$. Each detour (through E, through C) can be taken on either passage, so each type gives 4 actual routes, for $4 \cdot 4 = 16$ routes. Thus, the correct answer is **D**.

13. The internal angles of quadrilateral $ABCD$ form an arithmetic progression. Triangles ABD and DCB are similar with $\angle DBA = \angle DCB$ and $\angle ADB = \angle CBD$. Moreover, the angles in each of these two triangles also form an arithmetic progression. In degrees, what is the largest possible sum of the two largest angles of $ABCD$?

A 210

B 220

C 230

D 240

E 250

Solution:

The angles of a triangle form an arithmetic progression exactly when the middle one is $60°$. With $\angle DBA = x$ and $\angle ADB = y$, the four angles of $ABCD$ are $x, y, 180 - y, 180 - x$, which must itself be an arithmetic progression. Combined with a $60°$ angle in the triangles, this forces either $x = 60$ or $x + y = 120$. Working through the cases, the possible angle sets are $(60, 80, 100, 120)$ and $(45, 75, 105, 135)$. The two largest angles sum to at most $105 + 135 = 240$. Thus, the correct answer is **D**.

14. Two non-decreasing sequences of nonnegative integers have different first terms. Each sequence has the property that each term beginning with the third is the sum of the previous two terms, and the seventh term of each sequence is N. What is the smallest possible value of N?

A 55

B 89

C 104

D 144

E 273

Solution:

A sequence starting a_1, a_2 has seventh term $5a_1 + 8a_2$. For the two sequences, $5a_1 + 8a_2 = 5b_1 + 8b_2$, so $5(b_1 - a_1) = 8(a_2 - b_2)$. Since $\gcd(5, 8) = 1$, we need $8 \mid b_1 - a_1$ and $5 \mid a_2 - b_2$. Taking $a_1 < b_1$ with nondecreasing terms gives $a_1 \leq b_1 - 8 \leq b_2 - 8 \leq a_2 - 13$. Choosing $a_1 = 0, b_1 = b_2 = 8, a_2 = 13$ yields $N = 5 \cdot 0 + 8 \cdot 13 = 104$. Thus, the correct answer is **C**.

15. The number 2013 is expressed in the form

$$2013 = \frac{a_1!\,a_2!\cdots a_m!}{b_1!\,b_2!\cdots b_n!},$$

where $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ are positive integers and $a_1 + b_1$ is as small as possible. What is $|a_1 - b_1|$?

A 1

B 2

C 3

D 4

E 5

Solution:

Since $2013 = 3 \cdot 11 \cdot 61$, the numerator needs a factorial at least $61!$ to supply the prime 61, so $a_1 \geq 61$. But $61!$ also has a factor of 59, which 2013 does not, so the denominator needs $b_1 \geq 59$. Thus $a_1 + b_1 \geq 120$, attained by $a_1 = 61, b_1 = 59$ via $2013 = \dfrac{61!\,11!\,3!}{59!\,10!\,5!}$. Then $|a_1 - b_1| = 2$. Thus, the correct answer is **B**.

16. Let $ABCDE$ be an equiangular convex pentagon of perimeter 1. The pairwise intersections of the lines that extend the sides of the pentagon determine a five-pointed star polygon. Let s be the perimeter of this star. What is the difference between the maximum and the minimum possible values of s?

A 0

B $\frac{1}{2}$

C $\frac{\sqrt{5}-1}{2}$

D $\frac{\sqrt{5}+1}{2}$

E $\sqrt{5}$

Solution:

An equiangular pentagon has all interior angles 108°, so each point of the star is an isosceles triangle with base angles 72° and apex 36°. By the equal base angles, each point contributes two sides that are the same fixed multiple c of the pentagon side it rests on. Summing over the five points, the star perimeter equals $2c \cdot (\text{pentagon perimeter}) = 2c$, independent of the individual side lengths. So s is constant, and the difference between its maximum and minimum values is 0. Thus, the correct answer is **A**.

17. Let $a, b,$ and c be real numbers such that

$$a + b + c = 2 \quad \text{and} \quad a^2 + b^2 + c^2 = 12.$$

What is the difference between the maximum and minimum possible values of c?

A 2

B $\frac{10}{3}$

C 4

D $\frac{16}{3}$

E $\frac{20}{3}$

Solution:

From the equations, $a + b = 2 - c$ and $a^2 + b^2 = 12 - c^2$. Real numbers a, b with a given sum and sum of squares exist iff $(a+b)^2 \leq 2(a^2 + b^2)$, i.e. $(2-c)^2 \leq 2(12 - c^2)$. This simplifies to $(3c - 10)(c + 2) \leq 0$, so $-2 \leq c \leq \frac{10}{3}$. The difference is $\frac{10}{3} - (-2) = \frac{16}{3}$. Thus, the correct answer is **D**.

18. Barbara and Jenna play the following game, in which they take turns. A number of coins lie on a table. When it is Barbara's turn, she must remove 2 or 4 coins, unless only one coin remains, in which case she loses her turn. When it is Jenna's turn, she must remove 1 or 3 coins. A coin flip determines who goes first. Whoever removes the last coin wins the game. Assume both players use their best strategy. Who will win when the game starts with 2013 coins and when the game starts with 2014 coins?

A Barbara will win with 2013 coins, and Jenna will win with 2014 coins.

B Jenna will win with 2013 coins, and whoever goes first will win with 2014 coins.

C Barbara will win with 2013 coins, and whoever goes second will win with 2014 coins.

D Jenna will win with 2013 coins, and Barbara will win with 2014 coins.

E Whoever goes first will win with 2013 coins, and whoever goes second will win with 2014 coins.

Solution:

Work modulo 5. With $2013 \equiv 3$ coins, Jenna wins either way: going first she takes 3 to leave a multiple of 5, then answers Barbara's 2 with 3 and 4 with 1 to keep multiples of 5, eventually taking the last coin; going second she keeps the count $\equiv 3 \pmod 5$ until Barbara is stuck at 3 coins, must remove 2, and leaves Jenna the last coin. With $2014 \equiv 4$ coins, whoever goes first wins: Jenna first reduces to the 2013 case, while Barbara first takes 4 and then keeps multiples of 5. This is choice B. Thus, the correct answer is **B**.

19. In triangle $ABC, AB = 13, BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments BC, CA, and DE, respectively, such that $AD \perp BC, DE \perp AC$, and $AF \perp BF$. The length of segment DF can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

- A 18
- B 21
- C 24
- D 27
- E 30

Solution:

The altitude from A to BC gives $BD = 5, CD = 9, AD = 12$. Because $DE \perp AC$, triangle $AED \sim ADC$, giving $DE = \frac{36}{5}$ and $AE = \frac{48}{5}$. Since $\angle AFB = \angle ADB = 90^\circ$, quadrilateral $ABDF$ is cyclic, so $\angle ABD = \angle AFE$, making right triangles ABD and AFE similar: $\frac{FE}{5} = \frac{48/5}{12}$, so $FE = 4$. Hence $DF = DE - FE = \frac{36}{5} - 4 = \frac{16}{5}$, and $m + n = 21$. Thus, the correct answer is **B**.

20. For $135^\circ < x < 180^\circ$, points $P = (\cos x, \cos^2 x), Q = (\cot x, \cot^2 x), R = (\sin x, \sin^2 x)$, and $S = (\tan x, \tan^2 x)$ are the vertices of a trapezoid. What is $\sin(2x)$?

A $2 - 2\sqrt{2}$

B $3\sqrt{3} - 6$

C $3\sqrt{2} - 5$

D $-\frac{3}{4}$

E $1 - \sqrt{3}$

Solution:

Each point (t, t^2) lies on $y = t^2$, and the chord through parameters t_1, t_2 has slope $t_1 + t_2$. For $135^\circ < x < 180^\circ$, both $\cos x$ and $\tan x$ lie between $\cot x$ and $\sin x$, so P and S sit between Q and R and the parallel sides are QR and PS. Equal slopes give $\cot x + \sin x = \tan x + \cos x$. Multiplying by $\sin x \cos x$ and simplifying yields $\cos x + \sin x - \sin x \cos x = 0$. Squaring and using $2 \sin x \cos x = \sin 2x$ gives $1 + \sin 2x = \frac{1}{4} \sin^2 2x$, whose only root in $(-1, 1)$ is $\sin 2x = 2 - 2\sqrt{2}$. Thus, the correct answer is **A**.

21. Consider the set of 30 parabolas defined as follows: all parabolas have as focus the point $(0,0)$ and the directrix lines have the form $y = ax + b$ with a and b integers such that $a \in \{-2,-1,0,1,2\}$ and $b \in \{-3,-2,-1,1,2,3\}$. No three of these parabolas have a common point. How many points in the plane are on two of these parabolas?

A 720

B 760

C 810

D 840

E 870

Solution:

Two parabolas with common focus O meet in exactly 2 points, except when their directrices are parallel and O lies outside the strip between them, in which case they do not meet. The non-intersecting pairs have directrices of equal slope and y-intercepts of the same sign. There are 5 slopes, and for each, $2\binom{3}{2} = 6$ same-sign intercept pairs. Since every intersecting pair meets in 2 points and no point lies on three parabolas, the total is $2\left(\binom{30}{2} - 5 \cdot 6\right) = 2(435 - 30) = 810$. Thus, the correct answer is **C**.

22. Let $m > 1$ and $n > 1$ be integers. Suppose that the product of the solutions for x of the equation

$$8(\log_n x)(\log_m x) - 7\log_n x - 6\log_m x - 2013 = 0$$

is the smallest possible integer. What is $m + n$?

A 12

B 20

C 24

D 48

E 272

Solution:

Writing $\log_n x = \frac{\log x}{\log n}$ and $\log_m x = \frac{\log x}{\log m}$, the equation becomes a quadratic in $\log x$ whose roots sum to $\log(x_1 x_2) = \frac{1}{8}(7\log m + 6\log n)$. Hence $N^8 = m^7 n^6$, where $N = x_1 x_2$. For each prime dividing mn, the exponents a, b must satisfy $7a + 6b \equiv 0 \pmod 8$; minimizing the integer N gives $N = 16$, achieved uniquely at $m = 2^2 = 4$ and $n = 2^3 = 8$. So $m + n = 12$. Thus, the correct answer is **A**.

23. Bernardo chooses a three-digit positive integer N and writes both its base-5 and base-6 representations on a blackboard. Later LeRoy sees the two numbers Bernardo has written. Treating the two numbers as base-10 integers, he adds them to obtain an integer S. For example, if $N = 749$, Bernardo writes the numbers $10{,}444$ and $3{,}245$, and LeRoy obtains the sum $S = 13{,}689$. For how many choices of N are the two rightmost digits of S, in order, the same as those of $2N$?

- A 5
- B 10
- C 15
- D 20
- E 25

Solution:

Because $\text{lcm}(25, 36, 100) = 900$, the condition on N depends only on $N \bmod 900$, so consider $0 \leq N \leq 899$. Let the last two base-5 digits be a_1, a_0 and the last two base-6 digits be b_1, b_0. Matching the last two decimal digits of S and $2N$ forces the units digits equal, $a_0 = b_0$, and then working modulo 100 gives exactly 5 valid pairs $(a_1, b_1) : (0, 0), (2, 0), (4, 0), (1, 5)$, and $(3, 5)$. Each combines with 5 choices of a_0 $(0 \leq a_0 \leq 4)$, giving 25 values of N. Thus, the correct answer is **E**.

24. Let ABC be a triangle where M is the midpoint of AC, and CN is the angle bisector of $\angle ACB$ with N on AB. Let X be the intersection of the median BM and the bisector CN. In addition $\triangle BXN$ is equilateral and $AC = 2$. What is BN^2?

A $\dfrac{10-6\sqrt{2}}{7}$

B $\dfrac{2}{9}$

C $\dfrac{5\sqrt{2}-3\sqrt{3}}{8}$

D $\dfrac{\sqrt{2}}{6}$

E $\dfrac{3\sqrt{3}-4}{5}$

Solution:

Let $\alpha = \angle ACN = \angle NCB$ and $x = BN$. Since $\triangle BXN$ is equilateral, $\angle BXC = \angle CNA = 120°$, which gives $\triangle ABC \sim \triangle BMC$ and $\triangle ANC \sim \triangle BXC$. From the first, with $MC = \frac{1}{2}AC = 1$, we get $\dfrac{BC}{2} = \dfrac{MC}{BC}$, so $BC = \sqrt{2}$. From the second, $CX = (\sqrt{2}+1)x$. The Law of Cosines in $\triangle BCX$ with $\angle BXC = 120°$ gives $2 = x^2 + (\sqrt{2}+1)^2x^2 + (\sqrt{2}+1)x^2 = (5+3\sqrt{2})x^2$. Hence $BN^2 = x^2 = \dfrac{2}{5+3\sqrt{2}} = \dfrac{10-6\sqrt{2}}{7}$. Thus, the correct answer is **A**.

25. Let G be the set of polynomials of the form

$$P(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_2z^2 + c_1z + 50,$$

where $c_1, c_2, \ldots, c_{n-1}$ are integers and $P(z)$ has n distinct roots of the form $a + ib$ with a and b integers. How many polynomials are in G?

A 288

B 528

C 576

D 992

E 1056

Solution:

Since the coefficients are real, nonreal roots occur in conjugate pairs, so $P(z)$ factors into distinct linear factors $(z - c)$ with $c \in \mathbb{Z}$ and quadratics $(z - (a + ib))(z - (a - ib)) = z^2 - 2az + (a^2 + b^2)$. Each factor's constant term divides 50. Counting basic factors of magnitude d (the solutions of $a^2 + b^2 = d$, plus the two linear $z \pm d$) gives $|B_1| = 3, |B_2| = 4, |B_5| = 6, |B_{10}| = 6, |B_{25}| = 7, |B_{50}| = 8$. Building the constant term 50 as a single factor or a product over complementary divisors, and accounting for the free presence of $z + 1$ and $z^2 + 1$ (with $z - 1$ forced by the sign of the remaining product), gives

$$2^2 \left(8 + 7 \cdot 4 + 6 \cdot 6 + 4\binom{6}{2}\right) = 4(8 + 28 + 36 + 60) = 528.$$

Thus, the correct answer is **B**.

Problems: https://live.poshenloh.com/past-contests/amc12/2013B