2013 AMC 12A Solutions

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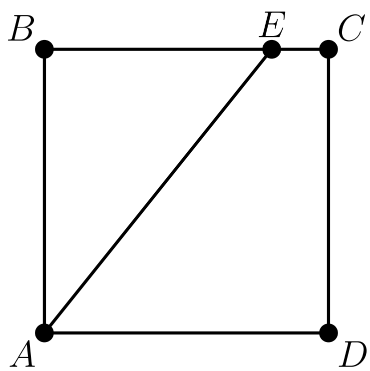
1. Square $ABCD$ has side length 10. Point E is on $\overline{BC}$, and the area of $\triangle ABE$ is 40. What is BE?



A 4

B 5

C 6

D 7

E 8

Solution:

The legs of right triangle ABE are $AB = 10$ and BE. From $\frac{1}{2} \cdot 10 \cdot BE = 40$, we get $BE = 8$.

Thus, the correct answer is **E**.

2. A softball team played ten games, scoring $1, 2, 3, 4, 5, 6, 7, 8, 9,$ and 10 runs. They lost by one run in exactly five games. In each of their other games, they scored twice as many runs as their opponent. How many total runs did their opponents score?

- A 35
- B 40
- C 45
- D 50
- E 55

Solution:

The team can only score twice as many runs as its opponent when its own score is even. Those games have scores $2, 4, 6, 8, 10,$ so their opponents scored $1 + 2 + 3 + 4 + 5 = 15.$

The other five games had scores $1, 3, 5, 7, 9$ and were one-run losses, so their opponents scored $2 + 4 + 6 + 8 + 10 = 30.$ The total is $15 + 30 = 45.$

Thus, the correct answer is **C**.

3. A flower bouquet contains pink roses, red roses, pink carnations, and red carnations. One third of the pink flowers are roses, three fourths of the red flowers are carnations, and six tenths of the flowers are pink. What percent of the flowers are carnations?

- A 15
- B 30
- C 40
- D 60
- E 70

Solution:

Six tenths of the flowers are pink and four tenths are red. Since two thirds of the pink flowers are carnations, pink carnations make up $\frac{2}{3} \cdot \frac{6}{10} = \frac{4}{10}$ of the flowers.

Since three fourths of the red flowers are carnations, red carnations make up $\frac{3}{4} \cdot \frac{4}{10} = \frac{3}{10}$ of the flowers. Together the carnations are $\frac{4}{10} + \frac{3}{10} = \frac{7}{10} = 70\%$.

Thus, the correct answer is **E**.

4. What is the value of

$$\frac{2^{2014}+2^{2012}}{2^{2014}-2^{2012}}?$$

A -1

B 1

C $\frac{5}{3}$

D 2013

E 2^{4024}

Solution:

Factoring 2^{2012} from each term gives

$$\frac{2^{2012}(2^2+1)}{2^{2012}(2^2-1)}=\frac{4+1}{4-1}=\frac{5}{3}.$$

Thus, the correct answer is **C**.

5. Tom, Dorothy, and Sammy went on a vacation and agreed to split the costs evenly. During their trip Tom paid \$105, Dorothy paid \$125, and Sammy paid \$175. In order to share the costs equally, Tom gave Sammy t dollars, and Dorothy gave Sammy d dollars. What is $t - d$?

A 15

B 20

C 25

D 30

E 35

Solution:

The total spent was $105 + 125 + 175 = 405$, so each fair share is $\frac{1}{3} \cdot 405 = 135$ dollars.

Then $t = 135 - 105 = 30$ and $d = 135 - 125 = 10$, so $t - d = 30 - 10 = 20$.

Thus, the correct answer is **B**.

6. In a recent basketball game, Shenille attempted only three-point shots and two-point shots. She was successful on 20% of her three-point shots and 30% of her two-point shots. Shenille attempted 30 shots. How many points did she score?

- A 12
- B 18
- C 24
- D 30
- E 36

Solution:

If Shenille attempted x three-point shots and $30 - x$ two-point shots, she scored

$$0.2 \cdot 3 \cdot x + 0.3 \cdot 2 \cdot (30 - x) = 0.6x + 0.6(30 - x) = 0.6 \cdot 30 = 18$$

points.

Thus, the correct answer is **B**.

7. The sequence $S_1, S_2, S_3, \ldots, S_{10}$ has the property that every term beginning with the third is the sum of the previous two. That is,

$$S_n = S_{n-2} + S_{n-1} \text{ for } n \geq 3.$$

Suppose that $S_9 = 110$ and $S_7 = 42$. What is S_4?

- A 4
- B 6
- C 10
- D 12
- E 16

Solution:

Since $S_9 = S_7 + S_8$, we get $S_8 = 110 - 42 = 68$. Then $S_6 = S_8 - S_7 = 68 - 42 = 26$, $S_5 = S_7 - S_6 = 42 - 26 = 16$, and $S_4 = S_6 - S_5 = 26 - 16 = 10$.

Thus, the correct answer is **C**.

8. Given that x and y are distinct nonzero real numbers such that $x + \frac{2}{x} = y + \frac{2}{y}$, what is xy?

A $\frac{1}{4}$

B $\frac{1}{2}$

C 1

D 2

E 4

Solution:

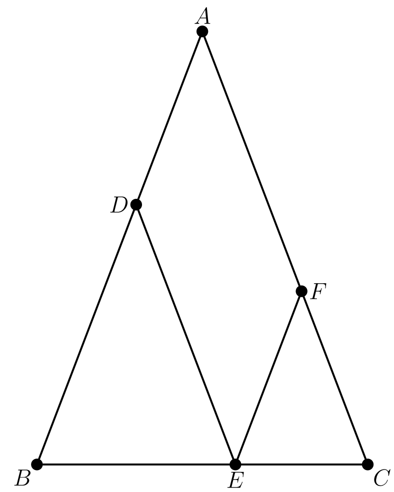
Multiplying by xy gives $x^2y + 2y = xy^2 + 2x$, so

$$x^2y - xy^2 - 2x + 2y = (x - y)(xy - 2) = 0.$$

Since $x \neq y$, it follows that $xy = 2.$

Thus, the correct answer is **D**.

9. In $\triangle ABC, AB = AC = 28$ and $BC = 20$. Points $D, E,$ and F are on sides $\overline{AB}, \overline{BC},$ and $\overline{AC}$, respectively, such that $\overline{DE}$ and $\overline{EF}$ are parallel to $\overline{AC}$ and $\overline{AB}$, respectively. What is the perimeter of parallelogram $ADEF$?



- A 48
- B 52
- C 56
- D 60
- E 72

Solution:

Because $EF \parallel AB$, triangle FEC is similar to triangle ABC, which is isosceles, so $FE = FC$.

Half the perimeter of parallelogram $ADEF$ is $AF + FE = AF + FC = AC = 28$. The entire perimeter is 56.

Thus, the correct answer is **C**.

10. Let S be the set of positive integers n for which $\frac{1}{n}$ has the repeating decimal representation $0.\overline{ab} = 0.ababab\ldots$, with a and b different digits. What is the sum of the elements of S?

A 11

B 44

C 110

D 143

E 155

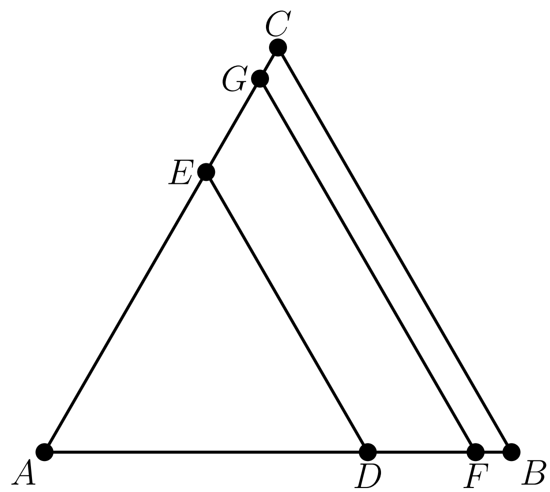
Solution:

If $\frac{1}{n} = 0.\overline{ab}$, then $\frac{99}{n} = \overline{ab}$, a two-digit number. The positive divisors of 99 are $1, 3, 9, 11, 33, 99.$

Only $n = 11, 33, 99$ make $\frac{99}{n}$ equal to $09, 03, 01,$ which have two different digits. The requested sum is $11 + 33 + 99 = 143.$

Thus, the correct answer is **D**.

11. Triangle ABC is equilateral with $AB = 1$. Points E and G are on $\overline{AC}$ and points D and F are on $\overline{AB}$ such that both $\overline{DE}$ and $\overline{FG}$ are parallel to $\overline{BC}$. Furthermore, triangle ADE and trapezoids $DFGE$ and $FBCG$ all have the same perimeter. What is $DE + FG$?



A 1

B $\frac{3}{2}$

C $\frac{21}{13}$

D $\frac{13}{8}$

E $\frac{5}{3}$

Solution:

Let $x = DE$ and $y = FG$. The parallel cuts make the small regions equilateral or isosceles trapezoids, so the perimeters are

$$\triangle ADE : 3x, \quad DFGE : 3y - x, \quad FBCG : 3 - y.$$

Setting them equal, $3x = 3y - x$ gives $4x = 3y$, and $3x = 3 - y$. Solving yields $x = \frac{9}{13}$ and $y = \frac{12}{13}$, so $DE + FG = \frac{21}{13}$.

Thus, the correct answer is **C**.

12. The angles in a particular triangle are in arithmetic progression, and the side lengths are $4, 5,$ and x. The sum of the possible values of x equals $a + \sqrt{b} + \sqrt{c}$, where $a, b,$ and c are positive integers. What is $a + b + c$?

A 36

B 38

C 40

D 42

E 44

Solution:

If the angles are $\alpha - \delta, \alpha, \alpha + \delta$, their sum $3\alpha = 180^\circ$ gives $\alpha = 60^\circ$, so one angle is 60°.

If x is opposite the 60° angle, the Law of Cosines gives

$$x^2 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cos 60^\circ = 21,$$

so $x = \sqrt{21}$.

If 5 is opposite the 60° angle, then $25 = x^2 - 4x + 16$, whose positive solution is $x = 2 + \sqrt{13}$. If 4 is opposite, then $16 = x^2 - 5x + 25$ has no real solution.

The sum of the possible values is $2 + \sqrt{13} + \sqrt{21}$, so $a + b + c = 2 + 13 + 21 = 36$.

Thus, the correct answer is **A**.

13. Let points $A = (0, 0), B = (1, 2), C = (3, 3)$, and $D = (4, 0)$. Quadrilateral $ABCD$ is cut into equal area pieces by a line passing through A. This line intersects $\overline{CD}$ at point $\left(\frac{p}{q}, \frac{r}{s}\right)$, where these fractions are in lowest terms. What is $p + q + r + s$?

A 54

B 58

C 62

D 70

E 75

Solution:

By the shoelace formula, the area of $ABCD$ is $\frac{15}{2}$. Let the line meet $\overline{CD}$ at G. Triangle ADG must have area $\frac{15}{4}$.

Since $AD = 4$ lies on the x-axis, $\frac{1}{2} \cdot 4 \cdot y_G = \frac{15}{4}$ gives $y_G = \frac{15}{8}$. Line CD is $y = -3(x - 4)$, so $x_G = \frac{27}{8}$.

Then $p + q + r + s = 27 + 8 + 15 + 8 = 58.$

Thus, the correct answer is **B**.

14. The sequence

$$\log_{12} 162,\ \log_{12} x,\ \log_{12} y,\ \log_{12} z,\ \log_{12} 1250$$

is an arithmetic progression. What is x?

- A $125\sqrt{3}$
- B 270
- C $162\sqrt{5}$
- D 434
- E $225\sqrt{6}$

Solution:

Because the logarithms are in arithmetic progression, $162, x, y, z, 1250$ is a geometric sequence. Its common ratio r satisfies $162r^4 = 1250$, so $r^4 = \frac{625}{81}$ and $r = \frac{5}{3}$.

Therefore $x = 162 \cdot \frac{5}{3} = 270.$

Thus, the correct answer is **B**.

15. Rabbits Peter and Pauline have three offspring—Flopsie, Mopsie, and Cottontail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

- A 96
- B 108
- C 156
- D 204
- E 372

Solution:

If the two parents share a store, there are 4 choices for it, and each child must go to one of the other three stores: $4 \cdot 3^3 = 108$ ways.

If the parents go to different stores, there are $4 \cdot 3 = 12$ choices, and each child must go to one of the two remaining stores: $12 \cdot 2^3 = 96$ ways.

The total is $108 + 96 = 204$.

Thus, the correct answer is **D**.

16. $A, B,$ and C are three piles of rocks. The mean weight of the rocks in A is 40 pounds, the mean weight of the rocks in B is 50 pounds, the mean weight of the rocks in the combined piles A and B is 43 pounds, and the mean weight of the rocks in the combined piles A and C is 44 pounds. What is the greatest possible integer value for the mean in pounds of the rocks in the combined piles B and C?

- A 55
- B 56
- C 57
- D 58
- E 59

Solution:

Let a, b, c be the numbers of rocks in the piles. From $\dfrac{40a + 50b}{a + b} = 43$, we get $7b = 3a$, so $a = 7k$ and $b = 3k$.

Let μ_{BC} be the mean of B and C. Using the A, C mean 44 to express $\mu_C = \dfrac{28k + 44c}{c}$, we find $\mu_{BC} = \dfrac{178k + 44c}{3k + c}$, so $(\mu_{BC} - 44)c = (178 - 3\mu_{BC})k$.

Since B is heavier than A, the mean of B and C exceeds 44, forcing $178 - 3\mu_{BC} > 0$, i.e. $\mu_{BC} < \frac{178}{3} = 59\frac{1}{3}$. The value 59 is attainable, so the greatest integer mean is 59.

Thus, the correct answer is **E**.

17. A group of 12 pirates agree to divide a treasure chest of gold coins among themselves as follows. The kth pirate to take a share takes $\frac{k}{12}$ of the coins that remain in the chest. The number of coins initially in the chest is the smallest number for which this arrangement will allow each pirate to receive a positive whole number of coins. How many coins does the 12th pirate receive?

A 720

B 1296

C 1728

D 1925

E 3850

Solution:

For $1 \leq k \leq 11,$ the number of coins before the kth pirate takes a share is $\frac{12}{12-k}$ times the number afterward. So if n coins are left for the 12th pirate, the initial count is

$$\frac{12^{11}\,n}{11!} = \frac{2^{14} \cdot 3^7\,n}{5^2 \cdot 7 \cdot 11}.$$

The smallest n making this a positive integer is $5^2 \cdot 7 \cdot 11 = 1925,$ and one checks each earlier pirate then receives a whole number of coins. The 12th pirate receives 1925 coins.

Thus, the correct answer is **D**.

18. Six spheres of radius 1 are positioned so that their centers are at the vertices of a regular hexagon of side length 2. The six spheres are internally tangent to a larger sphere whose center is the center of the hexagon. An eighth sphere is externally tangent to the six smaller spheres and internally tangent to the larger sphere. What is the radius of this eighth sphere?

A $\sqrt{2}$

B $\frac{3}{2}$

C $\frac{5}{3}$

D $\sqrt{3}$

E 2

Solution:

Each small center is 2 from the center O, and the small spheres have radius 1, so the large sphere has radius 3. Let the eighth sphere have radius r and center G at distance x from O; then $x + r = 3$.

Since G is equidistant from two opposite hexagon vertices, GO is perpendicular to the line to a vertex, and the Pythagorean Theorem gives

$$(r+1)^2 = 2^2 + x^2 = 4 + (3-r)^2.$$

This simplifies to $2r + 1 = 13 - 6r$, so $r = \frac{3}{2}$.

Thus, the correct answer is **B**.

19. In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

A 11

B 28

C 33

D 61

E 72

Solution:

By the Power of a Point Theorem, $BC \cdot CX = AC^2 - AB^2$ where AB is the radius. Thus $BC \cdot CX = 97^2 - 86^2 = 2013.$

Since $BC = BX + CX$ and CX are integers, they are complementary factors of $2013 = 3 \cdot 11 \cdot 61.$ As $CX < BC < AB + AC = 183,$ the only possibility is $CX = 33$ and $BC = 61.$

Thus, the correct answer is **D**.

20. Let S be the set $\{1, 2, 3, \ldots, 19\}$. For $a, b \in S$, define $a \succ b$ to mean that either $0 < a - b \le 9$ or $b - a > 9$. How many ordered triples (x, y, z) of elements of S have the property that $x \succ y, y \succ z$, and $z \succ x$?

A 810

B 855

C 900

D 950

E 988

Solution:

Reading the elements modulo 19, the relation $a \succ b$ holds exactly when $0 < (a - b) \bmod 19 \le 9$.

There are 19 choices for x. Once x is fixed, take $y = x + i$ for some $1 \le i \le 9$. Then z must satisfy $x + 10 \le z \le x + 9 + i$, giving i choices.

The total is $19(1 + 2 + \cdots + 9) = 19 \cdot 45 = 855$.

Thus, the correct answer is **B**.

21. Consider

$$A = \log(2013 + \log(2012 + \log(2011 + \log(\cdots + \log(3 + \log 2) \cdots)))).$$

Which of the following intervals contains A?

A $(\log 2016, \log 2017)$

B $(\log 2017, \log 2018)$

C $(\log 2018, \log 2019)$

D $(\log 2019, \log 2020)$

E $(\log 2020, \log 2021)$

Solution:

Let $A_n = \log(n + \log((n-1) + \cdots + \log(3 + \log 2) \cdots))$. One checks $0 < A_n < 1$ for $2 \leq n \leq 9$, then $1 < A_n < 2$ for $10 \leq n \leq 98$, then $2 < A_n < 3$ for $99 \leq n \leq 997$, and $3 < A_n < 4$ for $998 \leq n \leq 9996$.

Hence $3 < A_{2012} < 4$, so $2016 < 2013 + A_{2012} < 2017$ and therefore $\log 2016 < A < \log 2017$.

Thus, the correct answer is **A**.

22. A palindrome is a nonnegative integer number that reads the same forwards and backwards when written in base 10 with no leading zeros. A 6-digit palindrome n is chosen uniformly at random. What is the probability that $\frac{n}{11}$ is also a palindrome?

A $\frac{8}{25}$

B $\frac{33}{100}$

C $\frac{7}{20}$

D $\frac{9}{25}$

E $\frac{11}{30}$

Solution:

Let $m = \frac{n}{11}$. A 4-digit m leads to a contradiction, so m is a 5-digit palindrome $\overline{abcba}$. Writing $n = 11m$, there are no carries exactly when $a + b \leq 9$ and $b + c \leq 9$, and only then is n a palindrome. The number of valid m is

$$\sum_{b=0}^{9}(10 - b)(9 - b) = 330.$$

There are $9 \cdot 10^2 = 900$ six-digit palindromes, so the probability is $\frac{330}{900} = \frac{11}{30}$.

Thus, the correct answer is **E**.

23. $ABCD$ is a square of side length $\sqrt{3}+1$. Point P is on $\overline{AC}$ such that $AP=\sqrt{2}$. The square region bounded by $ABCD$ is rotated 90° counterclockwise with center P, sweeping out a region whose area is $\frac{1}{c}(a\pi+b)$, where $a, b,$ and c are positive integers and $\gcd(a,b,c)=1$. What is $a+b+c$?

A 15

B 17

C 19

D 21

E 23

Solution:

Let A', B', C', D' be the images of the vertices under the rotation. The swept region decomposes into four circular sectors and four triangles.

Since $AP=\sqrt{2}$ and $PC=AC-AP=\sqrt{6}$, the sectors at A and C have areas $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. One finds $PB=2$, so the two 60° sectors along BC each have area $\frac{2\pi}{3}$. The four triangles together contribute $(\sqrt{3}-1)+(3-\sqrt{3})=2$.

The total area is

$$\frac{\pi}{2}+\frac{3\pi}{2}+2\cdot\frac{2\pi}{3}+2=\frac{10\pi+6}{3},$$

so $a+b+c=10+6+3=19$.

Thus, the correct answer is **C**.

24. Three distinct segments are chosen at random among the segments whose endpoints are the vertices of a regular 12-gon. What is the probability that the lengths of these three segments are the three side lengths of a triangle with positive area?

A $\frac{553}{715}$

B $\frac{443}{572}$

C $\frac{111}{143}$

D $\frac{81}{104}$

E $\frac{223}{286}$

Solution:

Inscribe the 12-gon in a unit circle. The segment lengths are $d_k = 2\sin(15k^\circ)$ for $1 \le k \le 6$, with 12 segments of each length $d_1, \ldots, d_5$ and 6 of length d_6.

Comparing sums, the forbidden index triples (a, b, c) with $d_a \le d_b \le d_c$ and $d_c \ge d_a + d_b$ are

$$(1,1,3), (1,1,4), (1,1,5), (1,1,6), (1,2,4), (1,2,5), (1,2,6), (1,3,5), (1,3,6), (2,2,6).$$

Counting the corresponding segment selections and dividing by $\binom{66}{3}$ gives a failure probability of $\frac{63}{286}$, so the answer is $1 - \frac{63}{286} = \frac{223}{286}$.

Thus, the correct answer is **E**.

25. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z^2 + iz + 1$. How many complex numbers z are there such that $\text{Im}(z) > 0$ and both the real and the imaginary parts of $f(z)$ are integers with absolute value at most 10?

A 399

B 401

C 413

D 431

E 441

Solution:

On the upper half-plane H, if $f(z_1) = f(z_2)$ then $(z_1 - z_2)(z_1 + z_2 + i) = 0$; since $\text{Im}(z_1), \text{Im}(z_2) > 0$, the factor $z_1 + z_2 + i \neq 0$, so f is one-to-one on H.

The image is $f(H) = \{w : \text{Re}(w) < (\text{Im}(w))^2 + 1\}$. Thus we count $w = a + ib$ with $a, b \in \mathbb{Z}, |a|, |b| \leq 10$, and $a < b^2 + 1$:

$$|S| = 21^2 - \sum_{b=-3}^{3} (10 - b^2) = 441 - 42 = 399.$$

Thus, the correct answer is **A**.

Problems: https://live.poshenloh.com/past-contests/amc12/2013A