2012 AMC 12B Solutions

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1. Each third-grade classroom at Pearl Creek Elementary has 18 students and 2 pet rabbits. How many more students than rabbits are there in all 4 of the third-grade classrooms?

A 48

B 56

C 64

D 72

E 80

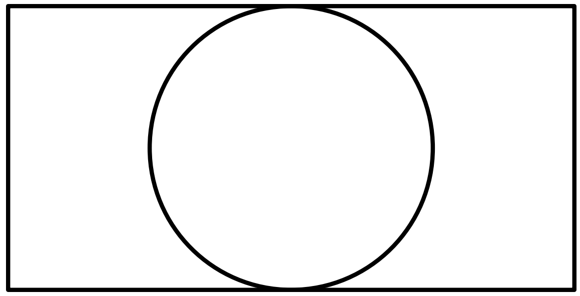
Solution:

Each classroom has $18 - 2 = 16$ more students than rabbits.

Across all 4 classrooms there are $4 \cdot 16 = 64$ more students than rabbits.

Thus, the correct answer is **C**.

2. A circle of radius 5 is inscribed in a rectangle as shown. The ratio of the length of the rectangle to its width is $2 : 1$. What is the area of the rectangle?



A 50

B 100

C 125

D 150

E 200

Solution:

The circle is inscribed, so the width of the rectangle equals the diameter, $2 \cdot 5 = 10.$

The length is then $2 \cdot 10 = 20,$ so the area is $10 \cdot 20 = 200.$

Thus, the correct answer is **E**.

3. For a science project, Sammy observed a chipmunk and a squirrel stashing acorns in holes. The chipmunk hid 3 acorns in each of the holes it dug. The squirrel hid 4 acorns in each of the holes it dug. They each hid the same number of acorns, although the squirrel needed 4 fewer holes. How many acorns did the chipmunk hide?

- A 30
- B 36
- C 42
- D 48
- E 54

Solution:

Let h be the number of holes the chipmunk dug. The chipmunk hid $3h$ acorns and the squirrel hid $4(h-4)$ acorns.

Since they hid the same number, $3h = 4(h-4)$, which gives $h = 16.$

The chipmunk hid $3 \cdot 16 = 48$ acorns.

Thus, the correct answer is **D**.

4. Suppose that the euro is worth 1.30 dollars. If Diana has 500 dollars and Étienne has 400 euros, by what percent is the value of Étienne's money greater than the value of Diana's money?

- A 2
- **B** 4
- C 6.5
- D 8
- E 13

Solution:

Étienne's money is worth $400 \cdot 1.30 = 520$ dollars, while Diana has 500 dollars.

The percent by which Étienne's value exceeds Diana's is

$$\frac{520 - 500}{500} \cdot 100\% = 4\%.$$

Thus, the correct answer is **B**.

5. Two integers have a sum of 26. When two more integers are added to the first two integers the sum is 41. Finally when two more integers are added to the sum of the previous four integers the sum is 57. What is the minimum number of even integers among the 6 integers?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

The three successive pairs have sums $26, 41 - 26 = 15$, and $57 - 41 = 16$.

A pair sums to an even number when its two integers share parity, and to an odd number when exactly one is even. Only the middle pair sums to an odd number, so it must contain at least one even integer.

The other two pairs can be all odd, so as few as 1 even integer is possible, for example $1, 25, 1, 14, 1, 15$.

Thus, the correct answer is **A**.

6. In order to estimate the value of $x - y$ where x and y are real numbers with $x > y > 0$, Xiaoli rounded x up by a small amount, rounded y down by the same amount, and then subtracted her rounded values. Which of the following statements is necessarily correct?

A Her estimate is larger than $x - y$.

B Her estimate is smaller than $x - y$.

C Her estimate equals $x - y$.

D Her estimate equals $y - x$.

E Her estimate is 0.

Solution:

Let $d > 0$ be the small amount. Xiaoli computes $(x + d) - (y - d) = (x - y) + 2d$.

Since $2d > 0$, her estimate exceeds the true value $x - y$.

Thus, the correct answer is **A**.

7. Small lights are hung on a string 6 inches apart in the order red, red, green, green, green, red, red, green, green, green, and so on continuing this pattern of 2 red lights followed by 3 green lights. How many feet separate the 3rd red light and the 21st red light?

Note: 1 foot is equal to 12 inches.

A 18

B 18.5

C 20

D 20.5

E 22.5

Solution:

The lights repeat in blocks of 5, so consecutive blocks start $5 \cdot 6 = 30$ inches, or 2.5 feet, apart.

Each block has one odd-numbered red light beginning it. The 3rd red light begins the 2nd block and the 21st red light begins the 11th block.

The distance between them is $(11 - 2) \cdot 2.5 = 22.5$ feet.

Thus, the correct answer is **E**.

8. A dessert chef prepares the dessert for every day of a week starting with Sunday. The dessert each day is either cake, pie, ice cream, or pudding. The same dessert may not be served two days in a row. There must be cake on Friday because of a birthday. How many different dessert menus for the week are possible?

A 729

B 972

C 1024

D 2187

E 2304

Solution:

Friday is fixed as cake. Work outward from Friday.

Each of the other six days (Saturday, then Thursday, Wednesday, Tuesday, Monday, Sunday) can be any dessert except the one served on the neighboring already-chosen day, giving 3 choices each.

The number of menus is $3^6 = 729.$

Thus, the correct answer is **A**.

9. It takes Clea 60 seconds to walk down an escalator when it is not operating, and only 24 seconds to walk down the escalator when it is operating. How many seconds does it take Clea to ride down the operating escalator when she just stands on it?

A 36

B 40

C 42

D 48

E 52

Solution:

Let x be Clea's walking rate and r the escalator's rate, with the length equal to $60x$. Walking on the moving escalator gives $24(x+r) = 60x$, so $r = \frac{3}{2}x$.

Standing takes time t with $rt = 60x$, so $\frac{3}{2}xt = 60x$ and $t = 40$ seconds.

Thus, the correct answer is **B**.

10. What is the area of the polygon whose vertices are the points of intersection of the curves $x^2 + y^2 = 25$ and $(x-4)^2 + 9y^2 = 81$?

A 24

B 27

C 36

D 37.5

E 42

Solution:

From $x^2 + y^2 = 25$ we get $y^2 = 25 - x^2$. Substituting into $(x-4)^2 + 9y^2 = 81$ gives $x^2 + x - 20 = 0$, so $x = 4$ or $x = -5$.

The intersection points are $(-5, 0), (4, 3)$, and $(4, -3)$.

The vertical side from $(4, 3)$ to $(4, -3)$ has length 6, and the horizontal distance to $(-5, 0)$ is 9, so the area is $\frac{1}{2} \cdot 6 \cdot 9 = 27$.

Thus, the correct answer is **B**.

11. In the equation below, A and B are consecutive positive integers, and $A, B,$ and $A + B$ represent number bases:

$$132_A + 43_B = 69_{A+B}.$$

What is $A + B$?

- A 9
- B 11
- C 13
- D 15
- E 17

Solution:

Writing the numerals out, $132_A = A^2 + 3A + 2, 43_B = 4B + 3,$ and $69_{A+B} = 6(A + B) + 9.$

With $B = A + 1,$ the equation becomes $A^2 + 3A + 2 + 4(A + 1) + 3 = 6(2A + 1) + 9,$ which simplifies to $(A - 6)(A + 1) = 0.$ The positive solution is $A = 6,$ so $B = 7.$

(The case $B = A - 1$ gives $A^2 - 5A - 2 = 0,$ which has no integer solution.)

Therefore $A + B = 13.$

Thus, the correct answer is **C**.

12. How many sequences of zeros and/or ones of length 20 have all the zeros consecutive, or all the ones consecutive, or both?

A 190

B 192

C 211

D 380

E 382

Solution:

Let A be the sequences in which all zeros are consecutive and B those in which all ones are consecutive.

For A, there is one all-ones sequence, 20 sequences with exactly one zero, and $\binom{20}{2} = 190$ sequences with two or more zeros (choose the first and last zero position). So $|A| = 1 + 20 + 190 = 211$, and by symmetry $|B| = 211$.

A sequence in $A \cap B$ is a block of zeros followed by a block of ones, or the reverse; there are $|A \cap B| = 40$ of these.

Therefore $|A \cup B| = 211 + 211 - 40 = 382$.

Thus, the correct answer is **E**.

13. Two parabolas have equations $y = x^2 + ax + b$ and $y = x^2 + cx + d$, where $a, b, c,$ and d are integers (not necessarily different), each chosen independently by rolling a fair six-sided die. What is the probability that the parabolas have at least one point in common?

A $\frac{1}{2}$

B $\frac{25}{36}$

C $\frac{5}{6}$

D $\frac{31}{36}$

E 1

Solution:

The parabolas meet where $x^2 + ax + b = x^2 + cx + d$, i.e. $ax + b = cx + d$. This has no solution exactly when the lines are parallel and distinct: $a = c$ and $b \neq d$.

The probability that $a = c$ is $\frac{1}{6}$, and the probability that $b \neq d$ is $\frac{5}{6}$, so the probability of no common point is $\frac{1}{6} \cdot \frac{5}{6} = \frac{5}{36}$.

The probability of at least one common point is $1 - \frac{5}{36} = \frac{31}{36}$.

Thus, the correct answer is **D**.

14. Bernardo and Silvia play the following game. An integer between 0 and 999, inclusive, is selected and given to Bernardo. Whenever Bernardo receives a number, he doubles it and passes the result to Silvia. Whenever Silvia receives a number, she adds 50 to it and passes the result to Bernardo. The winner is the last person who produces a number less than 1000. Let N be the smallest initial number that results in a win for Bernardo. What is the sum of the digits of N?

- A 7
- B 8
- C 9
- D 10
- E 11

Solution:

Bernardo wins after a round when his doubled number $2n + 50 \geq 1000$ but the previous numbers stayed below 1000. The smallest n with $2n + 50 \geq 1000$ is 475.

Working backwards, the smallest starting values that lead to a win after two, three, and four rounds are the smallest integers with $2n + 50 \geq 475, \geq 213,$ and $\geq 82,$ namely $213, 82,$ and $16.$ No start wins after more than four rounds.

So $N = 16,$ and the sum of its digits is $1 + 6 = 7.$

Thus, the correct answer is **A**.

15. Jesse cuts a circular paper disk of radius 12 along two radii to form two sectors, the smaller having a central angle of 120 degrees. He makes two circular cones, using each sector to form the lateral surface of a cone. What is the ratio of the volume of the smaller cone to that of the larger?

A $\frac{1}{8}$

B $\frac{1}{4}$

C $\frac{\sqrt{10}}{10}$

D $\frac{\sqrt{5}}{6}$

E $\frac{\sqrt{10}}{5}$

Solution:

Each sector forms a cone with slant height $12.$ The smaller sector's arc length is $\frac{120}{360} \cdot 2\pi \cdot 12 = 8\pi,$ so its base radius is 4 and its height is $\sqrt{12^2 - 4^2} = 8\sqrt{2}.$

The larger sector (central angle 240°) has arc length $16\pi,$ base radius $8,$ and height $\sqrt{12^2 - 8^2} = 4\sqrt{5}.$

The ratio of volumes is

$$\frac{\frac{1}{3}\pi \cdot 4^2 \cdot 8\sqrt{2}}{\frac{1}{3}\pi \cdot 8^2 \cdot 4\sqrt{5}} = \frac{\sqrt{10}}{10}.$$

Thus, the correct answer is **C**.

16. Amy, Beth, and Jo listen to four different songs and discuss which ones they like. No song is liked by all three. Furthermore, for each of the three pairs of the girls, there is at least one song liked by those two girls but disliked by the third. In how many different ways is this possible?

- A 108
- B 132
- C 671
- D 846
- E 1105

Solution:

Each song is liked by exactly one of the three pairs, by a single girl, or by no one. Every pair must be represented.

Case 1: every song is liked by a pair. One pair gets two of the four songs ($\binom{4}{2} = 6$ ways, and 3 choices for which pair), and the other two pairs get one song each (2 ways). This gives $3 \cdot 6 \cdot 2 = 36.$

Case 2: three songs go to the three pairs (one each) and the fourth song is liked by a single girl or no one. Assigning the four songs to these four roles gives $4! = 24$ ways, and the leftover role has 4 options (Amy, Beth, Jo, or no one): $24 \cdot 4 = 96.$

The total is $36 + 96 = 132.$

Thus, the correct answer is **B**.

17. Square $PQRS$ lies in the first quadrant. Points $(3,0),(5,0),(7,0)$, and $(13,0)$ lie on lines SP, RQ, PQ, and SR, respectively. What is the sum of the coordinates of the center of the square $PQRS$?

A 6

B 6.2

C 6.4

D 6.6

E 6.8

Solution:

Let θ be the acute angle line PQ makes with the x-axis. Sides $SR = PQ$ span the segment from $(3,0)$ to $(5,0)$ as $2\cos\theta$, while $SP = QR$ span the segment from $(7,0)$ to $(13,0)$ as $6\sin\theta$.

Since the square has equal sides, $2\cos\theta = 6\sin\theta$, so $\tan\theta = \frac{1}{3}$. Thus lines SP, RQ have slope 3 and lines SR, PQ have slope $-\frac{1}{3}$.

The center lies on the line through $(4,0)$ with slope 3 and the line through $(10,0)$ with slope $-\frac{1}{3}$:

$$y = 3(x-4), \qquad y = -\tfrac{1}{3}(x-10).$$

These meet at $(4.6, 1.8)$.

The sum of the coordinates is $4.6 + 1.8 = 6.4$.

Thus, the correct answer is **C**.

18. Let $(a_1, a_2, \ldots, a_{10})$ be a list of the first 10 positive integers such that for each $2 \leq i \leq 10$ either $a_i + 1$ or $a_i - 1$ or both appear somewhere before a_i in the list. How many such lists are there?

- A 120
- B 512
- C 1024
- D $181{,}440$
- E $362{,}880$

Solution:

Once $a_1 = k$ is fixed, the numbers $k, k+1, \ldots, 10$ must appear left to right in increasing order, and the numbers $1, \ldots, k-1$ must appear from right to left in increasing order (so each new small number has its successor already placed).

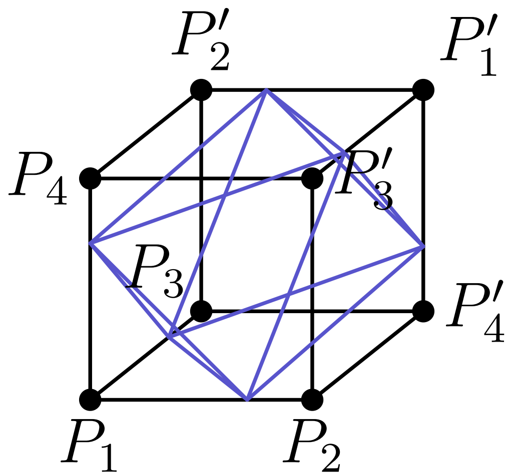
For each k, the list is determined by choosing which of the 9 positions after the first hold the numbers below k, giving $\binom{9}{k-1}$ lists.

Summing,

$$\sum_{k=1}^{10} \binom{9}{k-1} = \sum_{j=0}^{9} \binom{9}{j} = 2^9 = 512.$$

Thus, the correct answer is **B**.

19. A unit cube has vertices $P_1, P_2, P_3, P_4, P_1', P_2', P_3',$ and P_4'. Vertices $P_2, P_3,$ and P_4 are adjacent to P_1, and for $1 \le i \le 4$, vertices P_i and P_i' are opposite to each other. A regular octahedron has one vertex in each of the segments $P_1P_2, P_1P_3, P_1P_4, P_1'P_2',$ $P_1'P_3'$, and $P_1'P_4'$. What is the octahedron's side length?



A $\frac{3\sqrt{2}}{4}$

B $\frac{7\sqrt{6}}{16}$

C $\frac{\sqrt{5}}{2}$

D $\frac{2\sqrt{3}}{3}$

E $\frac{\sqrt{6}}{2}$

Solution:

Place P_1 at the origin with edges along the axes, and let each of the three octahedron vertices near P_1 be a distance t from P_1; by symmetry the three near P_1' are also a distance t from P_1'.

Two vertices sharing P_1, such as $(t, 0, 0)$ and $(0, t, 0)$, are a distance $t\sqrt{2}$ apart. A vertex near P_1, say $(t, 0, 0)$, and the appropriate vertex near P_1', say $(1, 1-t, 1)$, must be the same distance apart.

Setting the two squared side lengths equal and using the cube's unit edges yields $t = \frac{3}{4}$, so the side length is $t\sqrt{2} = \dfrac{3\sqrt{2}}{4}$.

Thus, the correct answer is **A**.

20. A trapezoid has side lengths $3, 5, 7,$ and 11. The sum of all the possible areas of the trapezoid can be written in the form of $r_1\sqrt{n_1} + r_2\sqrt{n_2} + r_3$, where $r_1, r_2,$ and r_3 are rational numbers and n_1 and n_2 are positive integers not divisible by the square of a prime. What is the greatest integer less than or equal to

$$r_1 + r_2 + r_3 + n_1 + n_2?$$

A 57

B 59

C 61

D 63

E 65

Solution:

For a trapezoid with parallel sides $a < c$ and legs b, d, translating a leg forms a triangle with sides $b, d,$ and $c - a$. The triangle inequality forces the longer parallel side to be $c = 11$.

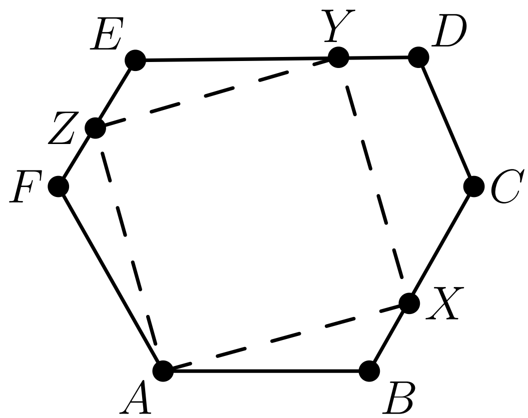
If $a = 3$, the triangle has sides $5, 7, 8$ with area $10\sqrt{3}$, and the trapezoid has area $\frac{35}{2}\sqrt{3}$. If $a = 5$, the triangle has sides $3, 6, 7$ with area $4\sqrt{5}$, giving trapezoid area $\frac{32}{3}\sqrt{5}$. If $a = 7$, the triangle has sides $3, 4, 5$, a right triangle, giving trapezoid area 27.

The total is $\frac{35}{2}\sqrt{3} + \frac{32}{3}\sqrt{5} + 27$, so $r_1 + r_2 + r_3 + n_1 + n_2 = \frac{35}{2} + \frac{32}{3} + 27 + 3 + 5 = 63 + \frac{1}{6}$.

The greatest integer at most this value is 63.

Thus, the correct answer is **D**.

21. Square $AXYZ$ is inscribed in equiangular hexagon $ABCDEF$ with X on $\overline{BC}$, Y on $\overline{DE}$, and Z on $\overline{EF}$. Suppose that $AB = 40$ and $EF = 41(\sqrt{3} - 1)$. What is the side-length of the square?



A $29\sqrt{3}$

B $\frac{21}{2}\sqrt{2} + \frac{41}{2}\sqrt{3}$

C $20\sqrt{3} + 16$

D $20\sqrt{2} + 13\sqrt{3}$

E $21\sqrt{6}$

Solution:

Extend EF and CB to a line through A perpendicular to both, meeting them at H and J. Since $\angle ABJ = 60^\circ$, we have $BJ = 20$ and $AJ = 20\sqrt{3}$. With $u = BX$, the Pythagorean theorem gives $s^2 = (20 + u)^2 + (20\sqrt{3})^2$.

The equiangular angles make the four corner triangles congruent, and chasing the equal segments along EF yields

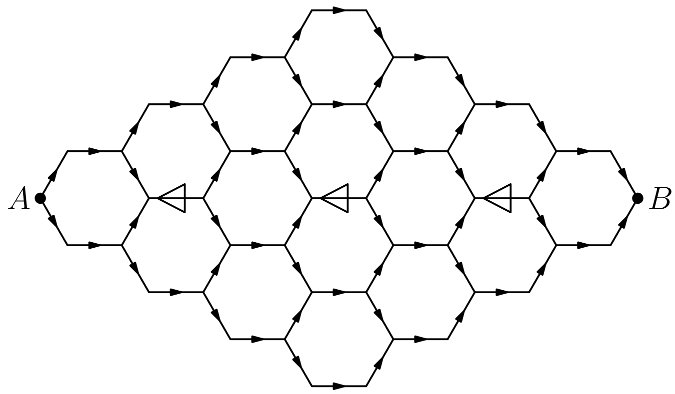
$$u + 20\sqrt{3} = 41(\sqrt{3} - 1) + \frac{20 + u}{\sqrt{3}},$$

so $u = 21\sqrt{3} - 20$.

Since $20 + u = 21\sqrt{3}$, we get $s^2 = (21\sqrt{3})^2 + (20\sqrt{3})^2 = 3(441 + 400) = 3 \cdot 29^2$, giving $s = 29\sqrt{3}$.

Thus, the correct answer is **A**.

22. A bug travels from A to B along the segments in the hexagonal lattice pictured below. The segments marked with an arrow can be traveled only in the direction of the arrow, and the bug never travels the same segment more than once. How many different paths are there?



A 2112

B 2304

C 2368

D 2384

E 2400

Solution:

Label the seven columns of forward (rightward) segments; a path with no back segment simply chooses one forward segment in each column. The numbers of choices are $2, 2, 4, 4, 4, 2, 2$, giving 2^{10} paths.

Let s_1, s_2, s_3 be the three left-pointing back segments (in columns $2, 4, 6$). Analyzing which columns become forced once a back segment is traversed gives 2^8 paths for each of $\{s_1\}$ and $\{s_3\}$, 2^6 for $\{s_1, s_3\}$, 2^9 for $\{s_2\}$, 2^7 for each of $\{s_1, s_2\}$ and $\{s_2, s_3\}$, and 2^5 for $\{s_1, s_2, s_3\}$.

Adding,

$$2^{10} + 2 \cdot 2^8 + 2^6 + 2^9 + 2 \cdot 2^7 + 2^5 = 2400.$$

Thus, the correct answer is **E**.

23. Consider all polynomials of a complex variable, $P(z) = 4z^4 + az^3 + bz^2 + cz + d$, where a, b, c, and d are integers, $0 \leq d \leq c \leq b \leq a \leq 4$, and the polynomial has a zero z_0 with $|z_0| = 1$. What is the sum of all values $P(1)$ over all the polynomials with these properties?

A 84

B 92

C 100

D 108

E 120

Solution:

Because $|z_0| = 1$, applying the triangle inequality to the identity $4z_0^5 - (z_0 - 1)P(z_0) = z_0^4(4-a) + z_0^3(a-b) + z_0^2(b-c) + z_0(c-d) + d$ forces equality throughout, so all but one of the coefficient differences vanish.

Working through the cases (including $z_0 = -1$ and $z_0 = \gamma$ a primitive cube root of unity), the polynomials are exactly $4z^4 + 4z^3 + 4z^2 + 4z + 4, 4z^4 + 4z^3 + 4z^2$, and $4z^4 + 4z^3 + bz^2 + bz$ for $0 \leq b \leq 4$.

Their values at 1 are $20, 12$, and $8 + 2b$; summing gives $20 + 12 + \sum_{b=0}^{4}(8 + 2b) = 32 + 40 + 20 = 92.$

Thus, the correct answer is **B**.

24. Define the function f_1 on the positive integers by setting $f_1(1) = 1$ and if $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime factorization of $n > 1$, then

$$f_1(n) = (p_1 + 1)^{e_1 - 1}(p_2 + 1)^{e_2 - 1} \cdots (p_k + 1)^{e_k - 1}.$$

For every $m \geq 2$, let $f_m(n) = f_1(f_{m-1}(n))$. For how many N in the range $1 \leq N \leq 400$ is the sequence $(f_1(N), f_2(N), f_3(N), \ldots)$ unbounded? Note: a sequence of positive numbers is unbounded if for every integer B, there is a member of the sequence greater than B.

A 15

B 16

C 17

D 18

E 19

Solution:

If $N_1 \mid N_2$ then $f_1(N_1) \mid f_1(N_2)$, so if S_{N_1} is unbounded so is S_{N_2}. Call N essential if it is unbounded but no proper divisor is. An essential N must have all exponents at least 2, and $(p_1 \cdots p_k)^2 \leq 400$ forces at most two primes.

Checking prime powers and prime pairs, the essential values are $2^5 = 32, 3^4 = 81$, $7^3 = 343$, and $2^4 \cdot 5^2 = 400$.

Their multiples up to 400 number $\lfloor 400/32 \rfloor = 12, \lfloor 400/81 \rfloor = 4, \lfloor 400/343 \rfloor = 1$, and $\lfloor 400/400 \rfloor = 1$, with no overlaps, for a total of $12 + 4 + 1 + 1 = 18$.

Thus, the correct answer is **D**.

25. Let $S = \{(x, y) : x \in \{0, 1, 2, 3, 4\}, y \in \{0, 1, 2, 3, 4, 5\}$, and $(x, y) \neq (0, 0)\}$. Let T be the set of all right triangles whose vertices are in S. For every right triangle $t = \triangle ABC$ with vertices A, B, and C in counter-clockwise order and right angle at A, let $f(t) = \tan(\angle CBA)$. What is

$$\prod_{t \in T} f(t)?$$

A 1

B $\dfrac{625}{144}$

C $\dfrac{125}{24}$

D 6

E $\dfrac{625}{24}$

Solution:

Isosceles right triangles contribute $f(t) = 1$. For a scalene right triangle, reflecting across a suitable line pairs it with a triangle t_1 so that $f(t)f(t_1) = \tan(\angle CBA)\tan(\angle ACB) = 1$.

Successive reflections (across $x = 2$, then $x = y$, then $y = \frac{5}{2}$) reduce the product to the reciprocal of the product over just six triangles of the form OYZ with Y on the top row.

Those six give

$$\frac{1}{5} \cdot \frac{2}{5} \cdot \frac{3}{5} \cdot \frac{4}{5} \cdot \frac{3\sqrt{2}}{\sqrt{2}} \cdot \frac{4\sqrt{2}}{\sqrt{2}} \cdot \frac{1}{2} = \frac{144}{625},$$

so the required product is its reciprocal, $\dfrac{625}{144}$.

Thus, the correct answer is **B**.

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