2012 AMC 12A Solutions

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1. A bug crawls along a number line, starting at $-2.$ It crawls to $-6,$ then turns around and crawls to $5.$ How many units does the bug crawl altogether?

A 9

B 11

C 13

D 14

E 15

Solution:

The bug crawls $|-6-(-2)|=4$ units on the first leg and $|5-(-6)|=11$ units on the second leg.

The total distance is $4+11=15.$

Thus, the correct answer is **E**.

2. Cagney can frost a cupcake every 20 seconds and Lacey can frost a cupcake every 30 seconds. Working together, how many cupcakes can they frost in 5 minutes?

A 10

B 15

C 20

D 25

E 30

Solution:

In 5 minutes there are 300 seconds. Cagney frosts $\frac{300}{20} = 15$ cupcakes and Lacey frosts $\frac{300}{30} = 10$ cupcakes.

Together they frost $15 + 10 = 25$ cupcakes.

Thus, the correct answer is **D**.

3. A box 2 centimeters high, 3 centimeters wide, and 5 centimeters long can hold 40 grams of clay. A second box with twice the height, three times the width, and the same length as the first box can hold n grams of clay. What is n?

- A 120
- B 160
- C 200
- D 240
- E 280

Solution:

The second box has $2 \times 3 \times 1 = 6$ times the volume of the first, so it holds 6 times as much clay.

Therefore $n = 6 \cdot 40 = 240.$

Thus, the correct answer is **D**.

4. In a bag of marbles, $\frac{3}{5}$ of the marbles are blue and the rest are red. If the number of red marbles is doubled and the number of blue marbles stays the same, what fraction of the marbles will be red?

A $\frac{2}{5}$

B $\frac{3}{7}$

C $\frac{4}{7}$

D $\frac{3}{5}$

E $\frac{4}{5}$

Solution:

Suppose there are 5 marbles: 3 blue and 2 red. Doubling the red gives 4 red while the blue stays at 3.

The total is now $3 + 4 = 7$, so the fraction that is red is $\frac{4}{7}$.

Thus, the correct answer is **C**.

5. A fruit salad consists of blueberries, raspberries, grapes, and cherries. The fruit salad has a total of 280 pieces of fruit. There are twice as many raspberries as blueberries, three times as many grapes as cherries, and four times as many cherries as raspberries. How many cherries are there in the fruit salad?

- A 8
- B 16
- C 25
- D 64
- E 96

Solution:

Let b be the number of blueberries. Then there are $2b$ raspberries, $4 \cdot 2b = 8b$ cherries, and $3 \cdot 8b = 24b$ grapes.

The total is $b + 2b + 8b + 24b = 35b = 280,$ so $b = 8$ and there are $8b = 64$ cherries.

Thus, the correct answer is **D**.

6. The sums of three whole numbers taken in pairs are $12, 17,$ and $19.$ What is the middle number?

- A 4
- B 5
- C 6
- D 7
- E 8

Solution:

Let the numbers be $a < b < c.$ Adding the three pairwise sums gives $2(a + b + c) = 12 + 17 + 19 = 48,$ so $a + b + c = 24.$

Then $a = 24 - 19 = 5, b = 24 - 17 = 7,$ and $c = 24 - 12 = 12.$ The middle number is $7.$

Thus, the correct answer is **D**.

7. Mary divides a circle into 12 sectors. The central angles of these sectors, measured in degrees, are all integers and they form an arithmetic sequence. What is the degree measure of the smallest possible sector angle?

A 5

B 6

C 8

D 10

E 12

Solution:

Let a be the smallest angle and $d \geq 0$ the common difference. The sum of the angles is $12a + 66d = 360$, so $2a + 11d = 60$.

To make a small, take d large. Since $11d$ must be even, d is even, and $d = 4$ gives $2a = 60 - 44 = 16$, so $a = 8$. A larger even d makes a non-positive.

Thus, the correct answer is **C**.

8. An *iterative average* of the numbers $1, 2, 3, 4,$ and 5 is computed in the following way. Arrange the five numbers in some order. Find the mean of the first two numbers, then find the mean of that with the third number, then the mean of that with the fourth number, and finally the mean of that with the fifth number. What is the difference between the largest and smallest possible values that can be obtained using this procedure?

A $\frac{31}{16}$

B 2

C $\frac{17}{8}$

D 3

E $\frac{65}{16}$

Solution:

For the order $a, b, c, d, e,$ the iterative average is

$$\frac{a+b+2c+4d+8e}{16}.$$

The later positions carry the most weight.

The largest value uses $(a, b, c, d, e) = (1, 2, 3, 4, 5)$, giving $\frac{65}{16}$, and the smallest uses $(5, 4, 3, 2, 1)$, giving $\frac{31}{16}$.

The difference is $\frac{65}{16} - \frac{31}{16} = \frac{34}{16} = \frac{17}{8}$.

Thus, the correct answer is **C**.

9. A year is a leap year if and only if the year number is divisible by 400 (such as 2000) or is divisible by 4 but not by 100 (such as 2012). The 200th anniversary of the birth of novelist Charles Dickens was celebrated on February $7, 2012,$ a Tuesday. On what day of the week was Dickens born?

- A Friday
- B Saturday
- C Sunday
- D Monday
- E Tuesday

Solution:

From February $7, 1812$ to February $7, 2012$ there are $200 \cdot 365 = 73000$ ordinary days plus one for each leap day.

One quarter of the 200 years contain a leap day, except $1900,$ giving $\frac{1}{4} \cdot 200 - 1 = 49$ leap days. So the span is 73049 days.

Since $73049 = 7 \cdot 10435 + 4,$ the birth day was 4 days before a Tuesday, which is a Friday.

Thus, the correct answer is **A**.

10. A triangle has area 30, one side of length 10, and the median to that side of length 9. Let θ be the acute angle formed by that side and the median. What is $\sin\theta$?

A $\frac{3}{10}$

B $\frac{1}{3}$

C $\frac{9}{20}$

D $\frac{2}{3}$

E $\frac{9}{10}$

Solution:

The median divides the triangle into two triangles of equal area 15. One of them has the two sides of length 5 (half the base) and 9 (the median) meeting at angle θ.

Its area is $\frac{1}{2}\cdot 5\cdot 9\sin\theta = 15$, so $\sin\theta = \dfrac{2\cdot 15}{5\cdot 9} = \dfrac{2}{3}$.

Thus, the correct answer is **D**.

11. Alex, Mel, and Chelsea play a game that has 6 rounds. In each round there is a single winner, and the outcomes of the rounds are independent. For each round the probability that Alex wins is $\frac{1}{2}$, and Mel is twice as likely to win as Chelsea. What is the probability that Alex wins three rounds, Mel wins two rounds, and Chelsea wins one round?

A $\frac{5}{72}$

B $\frac{5}{36}$

C $\frac{1}{6}$

D $\frac{1}{3}$

E 1

Solution:

Since Alex wins with probability $\frac{1}{2}$, the others share the remaining $\frac{1}{2}$. With Mel twice as likely as Chelsea, $P(\text{Mel}) = \frac{1}{3}$ and $P(\text{Chelsea}) = \frac{1}{6}$.

The number of orderings of the wins $AAAMMC$ is $\frac{6!}{3!\,2!\,1!} = 60$. The probability is

$$60 \cdot \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{3}\right)^2 \left(\tfrac{1}{6}\right) = \frac{60}{432} = \frac{5}{36}.$$

Thus, the correct answer is **B**.

12. A square region $ABCD$ is externally tangent to the circle with equation $x^2 + y^2 = 1$ at the point $(0, 1)$ on the side CD. Vertices A and B are on the circle with equation $x^2 + y^2 = 4$. What is the side length of this square?

A $\dfrac{\sqrt{10}+5}{10}$

B $\dfrac{2\sqrt{5}}{5}$

C $\dfrac{2\sqrt{2}}{3}$

D $\dfrac{2\sqrt{19}-4}{5}$

E $\dfrac{9-\sqrt{17}}{5}$

Solution:

By symmetry let $A = (a, b)$ with $a > 0$ and $B = (-a, b)$. The square sits on the tangent point $(0, 1)$, so its horizontal width is $2a$ and its height is $b - 1$.

Since these are equal, $2a = b - 1$, giving $b = 2a + 1$.

Substituting into $a^2 + b^2 = 4$ yields $5a^2 + 4a - 3 = 0$. The positive root is $a = \dfrac{\sqrt{19}-2}{5}$, so the side length is $2a = \dfrac{2\sqrt{19}-4}{5}$.

Thus, the correct answer is **D**.

13. Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 PM. How long, in minutes, was each day's lunch break?

- A 30
- B 36
- C 42
- D 48
- E 60

Solution:

Let m be the lunch length in minutes. The three worked $480 - m$ minutes Monday, the helpers $372 - m$ minutes Tuesday, and Paula $672 - m$ minutes Wednesday.

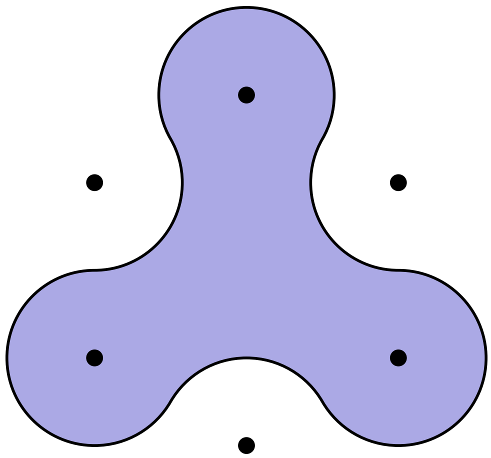
If Paula paints $p\%$ per minute and the helpers together paint $h\%$ per minute, then

$$(p+h)(480-m) = 50, \quad h(372-m) = 24, \quad p(672-m) = 26.$$

Adding the last two equations and subtracting from the first gives $108h - 192p = 0$, so $h = \frac{16}{9}p$. Solving the system gives $p = \frac{1}{24}$ and $m = 48$.

Thus, the correct answer is **D**.

14. The closed curve in the figure is made up of 9 congruent circular arcs each of length $\frac{2\pi}{3}$, where each of the centers of the corresponding circles is among the vertices of a regular hexagon of side 2. What is the area enclosed by the curve?



A $2\pi + 6$

B $2\pi + 4\sqrt{3}$

C $3\pi + 4$

D $2\pi + 3\sqrt{3} + 2$

E $\pi + 6\sqrt{3}$

Solution:

Each arc has length $\frac{2\pi}{3}$ on a unit circle, so it is a 120° sector. The nine equal sectors can be reassembled so that the enclosed region equals the regular hexagon of side 2 plus one full circle of radius 1.

A regular hexagon of side 2 splits into 6 equilateral triangles of side 2, so its area is $6 \cdot \frac{\sqrt{3}}{4} \cdot 2^2 = 6\sqrt{3}$.

Adding the unit circle's area π gives $\pi + 6\sqrt{3}$.

Thus, the correct answer is **E**.

15. A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random. The square is then rotated 90° clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability that the grid is now entirely black?

A $\frac{49}{512}$

B $\frac{7}{64}$

C $\frac{121}{1024}$

D $\frac{81}{512}$

E $\frac{9}{32}$

Solution:

The four corners form one cycle under the rotation, the four edge squares form another, and the center is fixed. These three groups are independent.

For the four corners, checking the $2^4 = 16$ colorings shows that 7 of them end all black, so the corners are all black with probability $\frac{7}{16}$. The same holds for the four edges.

The center is black at the end only if it started black, with probability $\frac{1}{2}$. Multiplying, the whole grid is black with probability

$$\frac{1}{2} \cdot \left(\frac{7}{16}\right)^2 = \frac{49}{512}.$$

Thus, the correct answer is **A**.

16. Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13, OZ = 11,$ and $YZ = 7$. What is the radius of circle C_1?

A 5

B $\sqrt{26}$

C $3\sqrt{3}$

D $2\sqrt{7}$

E $\sqrt{30}$

Solution:

Let r be the radius of C_1, so $OX = OY = r$. These are equal chords of C_2, so they subtend equal angles at $Z : \angle XZO = \angle OZY$.

Applying the Law of Cosines to triangles XZO and YZO,

$$\frac{13^2 + 11^2 - r^2}{2 \cdot 13 \cdot 11} = \frac{7^2 + 11^2 - r^2}{2 \cdot 7 \cdot 11}.$$

Clearing denominators and solving gives $r^2 = 30$, so $r = \sqrt{30}$.

Thus, the correct answer is **E**.

17. Let S be a subset of $\{1, 2, 3, \dots, 30\}$ with the property that no pair of distinct elements in S has a sum divisible by 5. What is the largest possible size of S?

A 10

B 13

C 15

D 16

E 18

Solution:

Group $\{1, \dots, 30\}$ by residue modulo 5; each class has 6 numbers. A sum is divisible by 5 when the residues are $0+0$, $1+4$, or $2+3$.

So S can use at most one number $\equiv 0$, and only one of the classes $\{1\}, \{4\}$ and only one of $\{2\}, \{3\}$. That allows at most $1 + 6 + 6 = 13$ numbers.

The set $\{1, 2, 6, 7, 11, 12, 16, 17, 21, 22, 26, 27, 30\}$ achieves 13, so the maximum is 13.

Thus, the correct answer is **B**.

18. Triangle ABC has $AB = 27, AC = 26,$ and $BC = 25.$ Let I denote the intersection of the internal angle bisectors of $\triangle ABC.$ What is BI?

A 15

B $5 + \sqrt{26} + 3\sqrt{3}$

C $3\sqrt{26}$

D $\frac{2}{3}\sqrt{546}$

E $9\sqrt{3}$

Solution:

Let D be the foot of the perpendicular from the incenter I to $BC.$ The tangent length $BD = s - AC,$ where $s = \frac{1}{2}(25 + 26 + 27) = 39,$ so $BD = 39 - 26 = 13.$

By Heron's formula the area is $\sqrt{39 \cdot 14 \cdot 13 \cdot 12},$ and the inradius satisfies $r^2 = \dfrac{(s-a)(s-b)(s-c)}{s} = \dfrac{14 \cdot 13 \cdot 12}{39} = 56.$

In right triangle $BDI, BI^2 = r^2 + BD^2 = 56 + 169 = 225,$ so $BI = 15.$

Thus, the correct answer is **A**.

19. Adam, Benin, Chiang, Deshawn, Esther, and Fiona have internet accounts. Some, but not all, of them are internet friends with each other, and none of them has an internet friend outside this group. Each of them has the same number of internet friends. In how many different ways can this happen?

A 60

B 170

C 290

D 320

E 660

Solution:

Model people as vertices of a graph, with edges for friendships. Everyone has the same degree n with $1 \leq n \leq 4$. The cases n and $6-1-n$ are complementary graphs, so $n=1$ pairs with $n=4$ and $n=2$ with $n=3$.

For $n=1$ the graph is a perfect matching: $5 \cdot 3 = 15$ ways. Thus $n=4$ also gives 15.

For $n=2$ the graph is a union of cycles: either two triangles $\left(\binom{5}{2}=10\right)$ or one hexagon $\left(\frac{6!}{12}=60\right)$, totaling 70. Thus $n=3$ also gives 70.

The total is $15+15+70+70=170$.

Thus, the correct answer is **B**.

20. Consider the polynomial

$$P(x) = \prod_{k=0}^{10} \left(x^{2^k} + 2^k\right) = (x+1)(x^2+2)(x^4+4)\cdots(x^{1024}+1024).$$

The coefficient of x^{2012} is equal to 2^a. What is a?

- A 5
- B 6
- C 7
- D 10
- E 24

Solution:

Expanding the product, a term of degree 2012 comes from choosing x^{2^k} from some factors so that the exponents sum to 2012. Since powers of two are distinct, this corresponds to the binary representation $2012 = 11111011100_2$.

That representation is unique, so exactly one term gives x^{2012}, and its coefficient is the product of the constants 2^k from the remaining factors: those with $k \in \{0, 1, 5\}$.

The coefficient is $2^0 \cdot 2^1 \cdot 2^5 = 2^6$, so $a = 6$.

Thus, the correct answer is **B**.

21. Let $a, b,$ and c be positive integers with $a \geq b \geq c$ such that

$$a^2 - b^2 - c^2 + ab = 2011$$

and

$$a^2 + 3b^2 + 3c^2 - 3ab - 2ac - 2bc = -1997.$$

What is a?

A 249

B 250

C 251

D 252

E 253

Solution:

Adding the two equations gives $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 14$, that is,

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 14.$$

The only way to write 14 as a sum of three squares is $9 + 4 + 1$. Since $a \geq b \geq c$, we get $a - c = 3$, with either $(a-b, b-c) = (2, 1)$ or $(1, 2)$.

Substituting $(a, b, c) = (c+3, c+1, c)$ into the first equation gives $3(2c+3) + 2(c+1) = 2011$, so $c = 250$ and $(a, b, c) = (253, 251, 250)$. The other case has no integer solution.

Thus, the correct answer is **E**.

22. Distinct planes $p_1, p_2, \ldots, p_k$ intersect the interior of a cube Q. Let S be the union of the faces of Q and let $P = \bigcup_{j=1}^{k} p_j$. The intersection of P and S consists of the union of all segments joining the midpoints of every pair of edges belonging to the same face of Q. What is the difference between the maximum and the minimum possible values of k?

A 8

B 12

C 20

D 23

E 24

Solution:

On every face, the required segments join midpoints of edges. A plane cutting the cube meets the faces in one of four symmetric shapes: a square through midpoints (3 such planes), a rectangle per edge (12 planes), a triangle per vertex (8 planes), or a regular hexagon per pair of opposite vertices (4 planes).

Using all of them gives the maximum $k = 3 + 12 + 8 + 4 = 27.$

The full figure consists of 24 short segments and 12 long segments. The 4 hexagon planes together contain all 24 short segments, and the 3 square planes contain all 12 long segments, so the minimum is $k = 4 + 3 = 7.$

The difference is $27 - 7 = 20.$

Thus, the correct answer is **C**.

23. Let S be the square one of whose diagonals has endpoints $(0.1, 0.7)$ and $(-0.1, -0.7)$. A point $v = (x, y)$ is chosen uniformly at random over all pairs of real numbers x and y such that $0 \leq x \leq 2012$ and $0 \leq y \leq 2012$. Let $T(v)$ be a translated copy of S centered at v. What is the probability that the square region determined by $T(v)$ contains exactly two points with integer coordinates in its interior?

A 0.125

B 0.14

C 0.16

D 0.25

E 0.32

Solution:

The diagonal from $(0.1, 0.7)$ to $(-0.1, -0.7)$ has length $\sqrt{0.2^2 + 1.4^2} = \sqrt{2}$, so S is a square of area 1. The translate $T(v)$ contains a lattice point exactly when v lies inside the copy of S centered at that point.

Containing exactly two interior lattice points requires v to lie in the overlap of two copies centered at adjacent lattice points. By periodicity the answer is the total such overlap area within one unit cell.

The overlap of two unit-area copies whose centers are one unit apart has area 0.08. Summing over the horizontal and vertical adjacencies gives probability $2 \cdot 0.08 = 0.16$.

Thus, the correct answer is **C**.

24. Let $\{a_k\}_{k=1}^{2011}$ be the sequence of real numbers defined by $a_1 = 0.201, a_2 = (0.2011)^{a_1}, a_3 = (0.20101)^{a_2}$, and $a_4 = (0.201011)^{a_3}$, and more generally

$$a_k = \begin{cases} \left(0.\underbrace{20101\ldots0101}_{k+2 \text{ digits}}\right)^{a_{k-1}}, & \text{if } k \text{ is odd,} \\ \left(0.\underbrace{20101\ldots01011}_{k+2 \text{ digits}}\right)^{a_{k-1}}, & \text{if } k \text{ is even.} \end{cases}$$

Rearranging the numbers in the sequence $\{a_k\}_{k=1}^{2011}$ in decreasing order produces a new sequence $\{b_k\}_{k=1}^{2011}$. What is the sum of all the integers $k, 1 \leq k \leq 2011$, such that $a_k = b_k$?

A 671

B 1006

C 1341

D 2011

E 2012

Solution:

Because each base lies strictly between 0 and 1, the function $t \mapsto (\text{base})^t$ is decreasing, while $t \mapsto t^b$ is increasing for $b > 0$. Comparing terms shows the sequence orders as

$$1 > a_2 > a_4 > \cdots > a_{2010} > a_{2011} > a_{2009} > \cdots > a_1 > 0.$$

So in the decreasing arrangement, the even-indexed terms come first, then the odd-indexed terms in reverse. A term satisfies $a_k = b_k$ exactly when its position equals its index, which for the descending odd tail requires $2(k - 1006) = 2011 - k$.

Solving gives $3k = 4023$, so $k = 1341$, the unique fixed index, and the sum is 1341.

Thus, the correct answer is **C**.

25. Let $f(x) = |2\{x\} - 1|$ where $\{x\}$ denotes the fractional part of x. The number n is the smallest positive integer such that the equation

$$nf(xf(x)) = x$$

has at least 2012 real solutions x. What is n?

Note: the fractional part of x is a real number $y = \{x\}$, such that $0 \leq y < 1$ and $x - y$ is an integer.

- A 30
- B 31
- C 32
- D 62
- E 64

Solution:

Since $0 \leq f(x) \leq 1$, every solution lies in $[0, n]$. The function f is a triangular wave of period 1, and $g(x) = xf(x)$ is monotonic on each half-integer interval, mapping it onto an interval on which $f(g(x))$ oscillates.

Counting the oscillations, on the intervals $[a, a + \frac{1}{2})$ and $[a + \frac{1}{2}, a + 1)$ the curve $y = f(g(x))$ meets the line $y = \frac{x}{n}$ a total of $2a$ and $2(a+1)$ times. Summing over $a = 0, \ldots, n-1$ gives

$$\sum_{a=0}^{n-1} \big(2a + 2(a+1)\big) = 2n^2$$

real solutions.

The smallest n with $2n^2 \geq 2012$ is $n = 32$, since $2 \cdot 31^2 = 1922$ and $2 \cdot 32^2 = 2048$.

Thus, the correct answer is **C**.

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