2011 AMC 12B Solutions

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1. What is

$$\frac{2+4+6}{1+3+5} - \frac{1+3+5}{2+4+6}?$$

A -1

B $\frac{5}{36}$

C $\frac{7}{12}$

D $\frac{147}{60}$

E $\frac{43}{3}$

Solution:

The sums are $2+4+6=12$ and $1+3+5=9,$ so the expression equals $\frac{12}{9} - \frac{9}{12} = \frac{4}{3} - \frac{3}{4}$.
Over a common denominator this is

$$\frac{16}{12} - \frac{9}{12} = \frac{7}{12}.$$

Thus, the correct answer is **C**.

2. Josanna's test scores to date are $90, 80, 70, 60,$ and $85.$ Her goal is to raise her test average at least 3 points with her next test. What is the minimum test score she would need to accomplish this goal?

- A 80
- B 82
- C 85
- D 90
- E 95

Solution:

The five scores sum to $90 + 80 + 70 + 60 + 85 = 385,$ giving an average of $77.$ The goal is a new average of at least $80.$

Six tests averaging 80 must total $6 \cdot 80 = 480,$ so the sixth score must be at least $480 - 385 = 95.$

Thus, the correct answer is **E**.

3. LeRoy and Bernardo went on a week-long trip together and agreed to share the costs equally. Over the week, each of them paid for various joint expenses such as gasoline and car rental. At the end of the trip it turned out that LeRoy had paid A dollars and Bernardo had paid B dollars, where $A < B$. How many dollars must LeRoy give to Bernardo so that they share the costs equally?

A $\dfrac{A+B}{2}$

B $\dfrac{A-B}{2}$

C $\dfrac{B-A}{2}$

D $B-A$

E $A+B$

Solution:

The total cost is $A+B$, so each person's fair share is $\dfrac{A+B}{2}$.

LeRoy paid A, which is less than his share, so he must give Bernardo

$$\frac{A+B}{2} - A = \frac{B-A}{2}.$$

Thus, the correct answer is **C**.

4. In multiplying two positive integers a and $b,$ Ron reversed the digits of the two-digit number $a.$ His erroneous product was $161.$ What is the correct value of the product of a and $b?$

A 116

B 161

C 204

D 214

E 224

Solution:

Since $161 = 7 \cdot 23,$ the only two-digit factor is $23.$ This must be the reversed value of $a,$ so the true value of a is $32,$ and $b = 7.$

The correct product is

$$32 \cdot 7 = 224.$$

Thus, the correct answer is **E**.

5. Let N be the second smallest positive integer that is divisible by every positive integer less than 7. What is the sum of the digits of N?

- A 3
- B 4
- C 5
- D 6
- E 9

Solution:

A number divisible by every integer from 1 to 6 must be a multiple of $\operatorname{lcm}(1, 2, 3, 4, 5, 6) = 60.$

The second smallest positive multiple of 60 is $120,$ whose digit sum is $1 + 2 + 0 = 3.$

Thus, the correct answer is **A**.

6. Two tangents to a circle are drawn from a point A. The points of contact B and C divide the circle into arcs with lengths in the ratio $2:3$. What is the degree measure of $\angle BAC$?

- A 24
- B 30
- C 36
- D 48
- E 60

Solution:

Let O be the center. The arcs measure $2x$ and $3x$ with $2x + 3x = 360°$, so $x = 72°$ and the minor arc BC gives central angle $\angle BOC = 144°$.

The radii to B and C are perpendicular to the tangents, so $\angle ABO = \angle ACO = 90°$. In quadrilateral $ABOC$,

$$\angle BAC = 360° - 144° - 90° - 90° = 36°.$$

Thus, the correct answer is **C**.

7. Let x and y be two-digit positive integers with mean 60. What is the maximum value of the ratio $\frac{x}{y}$?

- A 3
- B $\frac{33}{7}$
- C $\frac{39}{7}$
- D 9
- E $\frac{99}{10}$

Solution:

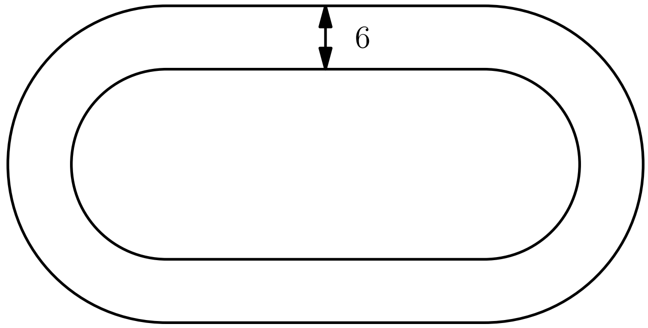
Since $\frac{x+y}{2} = 60,$ we have $x + y = 120.$ To maximize $\frac{x}{y}$ we make y small.

Because $x \leq 99,$ it follows that $y = 120 - x \geq 21.$ Taking $x = 99$ and $y = 21$ gives the maximum

$$\frac{99}{21} = \frac{33}{7}.$$

Thus, the correct answer is **B**.

8. Keiko walks once around a track at exactly the same constant speed every day. The sides of the track are straight, and the ends are semicircles. The track has width 6 meters, and it takes her 36 seconds longer to walk around the outside edge of the track than around the inside edge. What is Keiko's speed in meters per second?



A $\frac{\pi}{3}$

B $\frac{2\pi}{3}$

C π

D $\frac{4\pi}{3}$

E $\frac{5\pi}{3}$

Solution:

The straight sides are the same length for both paths, so the difference in length comes only from the two semicircular ends. If the inner radius is r, those ends combine into a full circle, and the extra length is

$$2\pi(r+6) - 2\pi r = 12\pi.$$

If her speed is x meters per second, then the extra time gives $36x = 12\pi$, so $x = \frac{\pi}{3}$.

Thus, the correct answer is **A**.

9. Two real numbers are selected independently at random from the interval $[-20, 10]$. What is the probability that the product of those numbers is greater than zero?

- A $\frac{1}{9}$
- B $\frac{1}{3}$
- C $\frac{4}{9}$
- D $\frac{5}{9}$
- E $\frac{2}{3}$

Solution:

The interval has length 30, with 20 of it negative and 10 of it positive. So each number is positive with probability $\frac{1}{3}$ and negative with probability $\frac{2}{3}$.

The product is positive when both are positive or both are negative:

$$\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}.$$

Thus, the correct answer is **D**.

10. Rectangle $ABCD$ has $AB = 6$ and $BC = 3.$ Point M is chosen on side AB so that $\angle AMD = \angle CMD.$ What is the degree measure of $\angle AMD$?

A 15

B 30

C 45

D 60

E 75

Solution:

Because $AB \parallel CD,$ we have $\angle CDM = \angle AMD.$ Combined with $\angle AMD = \angle CMD,$ this gives $\angle CDM = \angle CMD,$ so $\triangle CMD$ is isosceles with $CM = CD = 6.$

Then $\triangle MBC$ is right-angled at B with hypotenuse $CM = 6$ and leg $BC = 3,$ so it is a 30-60-90° triangle with $\angle BMC = 30^\circ.$

Finally, $\angle AMD + \angle CMD + \angle BMC = 180^\circ,$ so $2\angle AMD + 30^\circ = 180^\circ,$ giving $\angle AMD = 75^\circ.$

Thus, the correct answer is **E**.

11. A frog located at (x, y), with both x and y integers, makes successive jumps of length 5 and always lands on points with integer coordinates. Suppose that the frog starts at $(0, 0)$ and ends at $(1, 0)$. What is the smallest possible number of jumps the frog makes?

- A 2
- B 3
- C 4
- D 5
- E 6

Solution:

One jump cannot work, since $(0, 0)$ and $(1, 0)$ are only 1 apart. Two jumps also fail: the intermediate point would be at distance 5 from both, forcing it onto the perpendicular bisector $x = \frac{1}{2}$, which contains no lattice points.

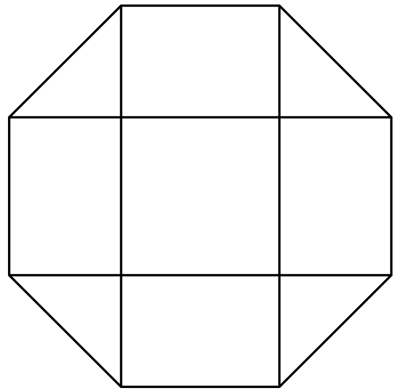
Three jumps suffice, for example

$$(0, 0) \to (3, 4) \to (6, 0) \to (1, 0),$$

where each step has length 5.

Thus, the correct answer is **B**.

12. A dart board is a regular octagon divided into regions as shown. Suppose that a dart thrown at the board is equally likely to land anywhere on the board. What is the probability that the dart lands within the center square?



A $\dfrac{\sqrt{2}-1}{2}$

B $\dfrac{1}{4}$

C $\dfrac{2-\sqrt{2}}{2}$

D $\dfrac{\sqrt{2}}{4}$

E $2-\sqrt{2}$

Solution:

Assume the octagon has edge length 1. The four corner triangles are right isosceles with legs $\dfrac{\sqrt{2}}{2}$ and area $\dfrac{1}{4}$ each. The four rectangles are 1 by $\dfrac{\sqrt{2}}{2}$ with area $\dfrac{\sqrt{2}}{2}$ each, and the center square has area 1.

The total area is

$$4 \cdot \frac{1}{4} + 4 \cdot \frac{\sqrt{2}}{2} + 1 = 2 + 2\sqrt{2}.$$

The probability of hitting the center square is

$$\frac{1}{2+2\sqrt{2}} = \frac{\sqrt{2}-1}{2}.$$

Thus, the correct answer is **A**.

13. Brian writes down four integers $w > x > y > z$ whose sum is 44. The pairwise positive differences of these numbers are $1, 3, 4, 5, 6,$ and 9. What is the sum of the possible values for w?

A 16

B 31

C 48

D 62

E 93

Solution:

The largest difference is $w - z = 9$. Writing $9 = (w-x)+(x-z)$ style splits, the interior differences pair as $3+6$ and $4+5$, which forces the smallest difference $x - y = 1$.

The second largest difference 6 is either $w - y$ or $x - z$. If $w - y = 6$, the numbers are $\{w, w-5, w-6, w-9\}$, so $4w - 20 = 44$ and $w = 16$. If $x - z = 6$, the numbers are $\{w, w-3, w-4, w-9\}$, so $4w - 16 = 44$ and $w = 15$.

The possible values are 16 and 15, which sum to 31.

Thus, the correct answer is **B**.

14. A segment through the focus F of a parabola with vertex V is perpendicular to $\overline{FV}$ and intersects the parabola in points A and B. What is $\cos(\angle AVB)$?

A $-\frac{3\sqrt{5}}{7}$

B $-\frac{2\sqrt{5}}{5}$

C $-\frac{4}{5}$

D $-\frac{3}{5}$

E $-\frac{1}{2}$

Solution:

Let $p = FV$ and let the directrix be ℓ. Projecting F and B onto ℓ, the focus-directrix property gives $FB = 2p$ (the distance from B to ℓ), and by the Pythagorean Theorem

$$VB = \sqrt{FV^2 + FB^2} = \sqrt{p^2 + 4p^2} = \sqrt{5}\,p.$$

Then $\cos(\angle FVB) = \dfrac{FV}{VB} = \dfrac{p}{\sqrt{5}\,p} = \dfrac{1}{\sqrt{5}}$. Since $\angle AVB = 2\angle FVB$,

$$\cos(\angle AVB) = 2\cos^2(\angle FVB) - 1 = 2 \cdot \frac{1}{5} - 1 = -\frac{3}{5}.$$

Thus, the correct answer is **D**.

15. How many positive two-digit integers are factors of $2^{24} - 1$?

A 4

B 8

C 10

D 12

E 14

Solution:

Factoring,

$$2^{24} - 1 = (2^{12} - 1)(2^{12} + 1) = (2^6 - 1)(2^6 + 1)(2^4 + 1)(2^8 - 2^4 + 1),$$

which equals $63 \cdot 65 \cdot 17 \cdot 241 = 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241.$

Since 241 is a three-digit prime, the two-digit factors come from $3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17.$ They are

$$13, 15, 17, 21, 35, 39, 45, 51, 63, 65, 85, 91,$$

for a total of $12.$

Thus, the correct answer is **D**.

16. Rhombus $ABCD$ has side length 2 and $\angle B = 120^\circ$. Region R consists of all points inside the rhombus that are closer to vertex B than any of the other three vertices. What is the area of R?

A $\frac{\sqrt{3}}{3}$

B $\frac{\sqrt{3}}{2}$

C $\frac{2\sqrt{3}}{3}$

D $1 + \frac{\sqrt{3}}{3}$

E 2

Solution:

Let E and H be the midpoints of AB and BC. The perpendicular bisector of AB through E meets diagonal AC at F, and the perpendicular bisector of BC through H meets AC at G. The region R is the pentagon $BEFGH$.

Triangle AFE is a $30\text{-}60\text{-}90^\circ$ triangle with $AE = 1$, so its area is $\frac{1}{2} \cdot 1 \cdot \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{6}$. Triangles BFE and BGH are congruent to it, and $\triangle FBG$ is equilateral, splitting into two more copies.

Hence R consists of four congruent triangles, giving area $4 \cdot \frac{\sqrt{3}}{6} = \frac{2\sqrt{3}}{3}$.

Thus, the correct answer is **C**.

17. Let $f(x) = 10^{10x}, g(x) = \log_{10}\left(\frac{x}{10}\right), h_1(x) = g(f(x))$, and $h_n(x) = h_1(h_{n-1}(x))$ for integers $n \geq 2$. What is the sum of the digits of $h_{2011}(1)$?

A $16{,}081$

B $16{,}089$

C $18{,}089$

D $18{,}098$

E $18{,}099$

Solution:

First,

$$h_1(x) = \log_{10}\left(\frac{10^{10x}}{10}\right) = \log_{10}\left(10^{10x-1}\right) = 10x - 1.$$

Iterating, $h_n(x) = 10^n x - (1 + 10 + \cdots + 10^{n-1})$. Therefore $h_n(1)$ is an n-digit integer whose units digit is 9 and all of whose other digits are 8.

For $n = 2011$, the digit sum is

$$8 \cdot 2010 + 9 = 16{,}089.$$

Thus, the correct answer is **B**.

18. A pyramid has a square base with sides of length 1 and has lateral faces that are equilateral triangles. A cube is placed within the pyramid so that one face is on the base of the pyramid and its opposite face has all its edges on the lateral faces of the pyramid. What is the volume of this cube?

A $5\sqrt{2} - 7$

B $7 - 4\sqrt{3}$

C $\frac{2\sqrt{2}}{27}$

D $\frac{\sqrt{2}}{9}$

E $\frac{\sqrt{3}}{9}$

Solution:

Let the apex be A and the base be square $BCDE.$ Then $AB = AD = 1$ and $BD = \sqrt{2},$ so $\triangle BAD$ is an isosceles right triangle.

Let the cube have edge length $x.$ Its intersection with the plane of $\triangle BAD$ is a rectangle of height x and width $\sqrt{2}\,x,$ whose top corners lie on AB and $AD.$ Because the legs AB and AD meet the base at 45°, each portion of BD outside the rectangle has length $x,$ so

$$\sqrt{2} = BD = \sqrt{2}\,x + 2x,$$

which reduces to $x = \dfrac{\sqrt{2}}{2+\sqrt{2}} = \sqrt{2} - 1.$

The volume is

$$(\sqrt{2} - 1)^3 = 5\sqrt{2} - 7.$$

Thus, the correct answer is **A**.

19. A lattice point in an xy-coordinate system is any point (x, y) where both x and y are integers. The graph of $y = mx + 2$ passes through no lattice point with $0 < x \leq 100$ for all m such that $\frac{1}{2} < m < a$. What is the maximum possible value of a?

A $\frac{51}{101}$

B $\frac{50}{99}$

C $\frac{51}{100}$

D $\frac{52}{101}$

E $\frac{13}{25}$

Solution:

For $0 < x \leq 100$, the nearest lattice point above the line $y = \frac{1}{2}x + 2$ is $\left(x, \frac{1}{2}x + 3\right)$ if x is even and $\left(x, \frac{1}{2}x + \frac{5}{2}\right)$ if x is odd.

The slope from $(0, 2)$ to that point is $\frac{1}{2} + \frac{1}{x}$ for even x and $\frac{1}{2} + \frac{1}{2x}$ for odd x. The minimum such slope is $\frac{51}{100}$ for even x and $\frac{50}{99}$ for odd x.

Since $\frac{50}{99} < \frac{51}{100}$, the line avoids all these lattice points exactly when $\frac{1}{2} < m < \frac{50}{99}$, so the maximum is $a = \frac{50}{99}$.

Thus, the correct answer is **B**.

20. Triangle ABC has $AB = 13, BC = 14,$ and $AC = 15.$ The points $D, E,$ and F are the midpoints of $AB, BC,$ and AC respectively. Let $X \neq E$ be the intersection of the circumcircles of $\triangle BDE$ and $\triangle CEF.$ What is $XA + XB + XC$?

A 24

B $14\sqrt{3}$

C $\frac{195}{8}$

D $\frac{129\sqrt{7}}{14}$

E $\frac{69\sqrt{2}}{4}$

Solution:

Since $DE \parallel AC$ and $EF \parallel AB,$ we get $\angle BDE = \angle BAC = \angle EFC.$ By the Inscribed Angle Theorem, $\angle BXE = \angle BDE$ and $\angle EXC = \angle EFC,$ so $\angle BXE = \angle EXC.$ With $BE = EC,$ this forces $XB = XC.$

Also $\angle BXC = 2\angle BAC,$ so by the Inscribed Angle Theorem X is the circumcenter of $\triangle ABC.$ Hence $XA = XB = XC = R.$

The area of the 13-14-15 triangle is 84 by Heron's formula, so

$$R = \frac{13 \cdot 14 \cdot 15}{4 \cdot 84} = \frac{65}{8},$$

and $XA + XB + XC = 3R = \frac{195}{8}.$

Thus, the correct answer is **C**.

21. The arithmetic mean of two distinct positive integers x and y is a two-digit integer. The geometric mean of x and y is obtained by reversing the digits of the arithmetic mean. What is $|x-y|$?

A 24

B 48

C 54

D 66

E 70

Solution:

Let the arithmetic mean be $10a+b$ and the geometric mean be $10b+a$. Then $x+y=2(10a+b)$ and $xy=(10b+a)^2$.

Therefore

$$(x-y)^2=(x+y)^2-4xy=396(a^2-b^2)=11\cdot 6^2\cdot(a+b)(a-b).$$

This is a perfect square exactly when $a+b=11$ and $a-b$ is a perfect square. Among digit solutions, only $a-b=1$ works, giving $(a,b)=(6,5)$.

Then $(x-y)^2=11\cdot 6^2\cdot 11=66^2$, so $|x-y|=66$. (Indeed $\{x,y\}=\{32,98\}$.)

Thus, the correct answer is **D**.

22. Let T_1 be a triangle with sides $2011, 2012$, and 2013. For $n \geq 1$, if $T_n = \triangle ABC$ and D, E, and F are the points of tangency of the incircle of $\triangle ABC$ to the sides AB, BC, and AC, respectively, then T_{n+1} is a triangle with side lengths AD, BE, and CF, if it exists. What is the perimeter of the last triangle in the sequence (T_n)?

- A $\frac{1509}{8}$
- B $\frac{1509}{32}$
- C $\frac{1509}{64}$
- D $\frac{1509}{128}$
- E $\frac{1509}{256}$

Solution:

For a triangle with sides a, b, c, the tangent lengths are $AD = \frac{1}{2}(b + c - a), BE = \frac{1}{2}(a + c - b)$, and $CF = \frac{1}{2}(a + b - c)$. If T_n has sides $(y - 1, y, y + 1)$, then T_{n+1} has sides $\left(\frac{y}{2} - 1, \frac{y}{2}, \frac{y}{2} + 1\right)$.

Starting from T_1 with middle side 2012, the middle side halves each step and the perimeter of T_{n+1} is $\frac{1}{2}$ the perimeter of T_n. A triangle of this form exists only while its middle side exceeds 2.

The middle side of T_n is $\dfrac{2012}{2^{n-1}}$. This first drops to 2 or below at $n = 11$, so the last valid triangle is T_{10}, whose middle side is $\dfrac{2012}{2^9}$ and whose perimeter is

$$3 \cdot \frac{2012}{2^9} = \frac{6036}{512} = \frac{1509}{128}.$$

Thus, the correct answer is **D**.

23. A bug travels in the coordinate plane, moving only along the lines that are parallel to the x-axis or y-axis. Let $A = (-3, 2)$ and $B = (3, -2)$. Consider all possible paths of the bug from A to B of length at most 20. How many points with integer coordinates lie on at least one of these paths?

- A 161
- B 185
- C 195
- D 227
- E 255

Solution:

A lattice point $X = (x, y)$ lies on some path exactly when

$$d = |x-3| + |x+3| + |y-2| + |y+2| \leq 20.$$

This expression is unchanged when $x \to -x$ or $y \to -y$, so we count points with $x \geq 0, y \geq 0$, multiply by 4, and correct for the axes.

Splitting into the four regions determined by whether $x \leq 3$ and $y \leq 2$ gives $12 + 20 + 15 + 10 = 57$ points in the first quadrant (including axis points). By symmetry the total is

$$4 \cdot 57 - 2 \cdot 15 - 3 = 195.$$

Thus, the correct answer is **C**.

24. Let $P(z) = z^8 + (4\sqrt{3} + 6)z^4 - (4\sqrt{3} + 7)$. What is the minimum perimeter among all the 8-sided polygons in the complex plane whose vertices are precisely the zeros of $P(z)$?

A $4\sqrt{3} + 4$

B $8\sqrt{2}$

C $3\sqrt{2} + 3\sqrt{6}$

D $4\sqrt{2} + 4\sqrt{3}$

E $4\sqrt{3} + 6$

Solution:

Factoring in z^4,

$$P(z) = (z^4 - 1)\big(z^4 + (4\sqrt{3} + 7)\big).$$

The first factor gives the roots $1, -1, i, -i$. Since $4\sqrt{3} + 7 = (\sqrt{3} + 2)^2$ and $2(\sqrt{3} + 2) = (\sqrt{3} + 1)^2$, writing $w = \frac{1}{2}(\sqrt{3} + 1)$ the other four roots are $w(\pm 1 \pm i)$.

The eight roots are symmetric about the origin with 4-fold symmetry, and every segment joining two of them has length at least $\sqrt{2}$. Thus any such polygon has perimeter at least $8\sqrt{2}$, and the polygon with vertices $1, w(1 + i), i, w(-1 + i), -1, w(-1 - i), -i, w(1 - i)$ achieves it.

Thus, the correct answer is **B**.

25. For every m and k integers with k odd, denote by $\left[\frac{m}{k}\right]$ the integer closest to $\frac{m}{k}$. For every odd integer k, let $P(k)$ be the probability that

$$\left[\frac{n}{k}\right] + \left[\frac{100-n}{k}\right] = \left[\frac{100}{k}\right]$$

for an integer n randomly chosen from the interval $1 \leq n \leq 99!$. What is the minimum possible value of $P(k)$ over the odd integers k in the interval $1 \leq k \leq 99$?

A $\frac{1}{2}$

B $\frac{50}{99}$

C $\frac{44}{87}$

D $\frac{34}{67}$

E $\frac{7}{13}$

Solution:

Because $\left[\frac{n+mk}{k}\right] = \left[\frac{n}{k}\right] + m$, whether n satisfies the identity depends only on $n \bmod k$. Since $k \mid 99!$ for $1 \leq k \leq 99$, every residue class is equally likely.

Write $100 = qk + r$ with $|r| \leq \frac{k-1}{2}$. Analyzing the carry in $[\,(100-n)/k\,]$ shows the identity holds precisely for the residues in an interval of the appropriate length, giving

$$P(k) = 1 - \frac{|r|}{k}.$$

To minimize $P(k)$ we maximize $\frac{|r|}{k}$. Taking $r = \frac{k-1}{2}$ gives $201 = k(2q+1)$, and the largest odd $k \leq 99$ dividing $201 = 3 \cdot 67$ is $k = 67$. Then

$$P(67) = \frac{1}{2} + \frac{1}{2 \cdot 67} = \frac{34}{67},$$

which is smaller than the values from all other cases.

Thus, the correct answer is **D**.

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