2011 AMC 12A Solutions

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1. A cell phone plan costs $\$20$ each month, plus $5¢$ per text message sent, plus $10¢$ for each minute used over 30 hours. In January Michelle sent 100 text messages and talked for 30.5 hours. How much did she have to pay?

A $\$24.00$

B $\$24.50$

C $\$25.50$

D $\$28.00$

E $\$30.00$

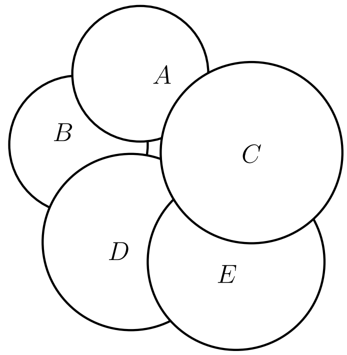
Solution:

The text charge is $100 \cdot 5 = 500$ cents $= \$5.$ She talked 30 minutes past the 30-hour allowance, so the overage is $30 \cdot 10 = 300$ cents $= \$3.$

The total is $\$20 + \$5 + \$3 = \$28.$

Thus, the correct answer is **D**.

2. There are 5 coins placed flat on a table according to the figure. What is the order of the coins from top to bottom?



A (C, A, E, D, B)

B (C, A, D, E, B)

C (C, D, E, A, B)

D (C, E, A, D, B)

E (C, E, D, A, B)

Solution:

Coin C is drawn as a complete, unbroken circle, so nothing covers it and it lies on top.

Reading the remaining overlaps, each coin's uncovered arc shows it sits above the next: C covers E, E covers D, D covers B, and A covers B while lying under the others. This gives the top-to-bottom order (C, E, D, A, B).

Thus, the correct answer is **E**.

3. A small bottle of shampoo can hold 35 milliliters of shampoo, whereas a large bottle can hold 500 milliliters of shampoo. Jasmine wants to buy the minimum number of small bottles necessary to completely fill a large bottle. How many bottles must she buy?

A 11

B 12

C 13

D 14

E 15

Solution:

Fourteen bottles hold $14 \cdot 35 = 490$ milliliters, which is not enough. Fifteen bottles hold $15 \cdot 35 = 525$ milliliters, which suffices.

So Jasmine needs 15 bottles.

Thus, the correct answer is **E**.

4. At an elementary school, the students in third grade, fourth grade, and fifth grade run an average of $12, 15,$ and 10 minutes per day, respectively. There are twice as many third graders as fourth graders, and twice as many fourth graders as fifth graders. What is the average number of minutes run per day by these students?

A 12

B $\frac{37}{3}$

C $\frac{88}{7}$

D 13

E 14

Solution:

Take the grade sizes in the ratio $4 : 2 : 1$ for third, fourth, and fifth grades. The weighted average is

$$\frac{4 \cdot 12 + 2 \cdot 15 + 1 \cdot 10}{7} = \frac{48 + 30 + 10}{7} = \frac{88}{7}.$$

Thus, the correct answer is **C**.

5. Last summer 30% of the birds living on Town Lake were geese, 25% were swans, 10% were herons, and 35% were ducks. What percent of the birds that were not swans were geese?

A 20

B 30

C 40

D 50

E 60

Solution:

The birds that are not swans make up $100\% - 25\% = 75\%$ of the total, and geese are 30% of the total. The requested fraction is

$$\frac{30}{75} = \frac{2}{5} = 40\%.$$

Thus, the correct answer is **C**.

6. The players on a basketball team made some three-point shots, some two-point shots, and some one-point free throws. They scored as many points with two-point shots as with three-point shots. Their number of successful free throws was one more than their number of successful two-point shots. The team's total score was 61 points. How many free throws did they make?

A 13

B 14

C 15

D 16

E 17

Solution:

Let a be the number of two-point shots. The two-point shots score $2a$ points, and the three-point shots score the same $2a$ points. The free throws number $a+1$ and score $a+1$ points.

The total is

$$2a + 2a + (a+1) = 5a + 1 = 61,$$

so $a = 12$ and the free throws number $a + 1 = 13.$

Thus, the correct answer is **A**.

7. A majority of the 30 students in Ms. Demeanor's class bought pencils at the school bookstore. Each of these students bought the same number of pencils, and this number was greater than $1.$ The cost of a pencil in cents was greater than the number of pencils each student bought, and the total cost of all the pencils was $\$17.71.$ What was the cost of a pencil in cents?

- A 7
- B 11
- C 17
- D 23
- E 77

Solution:

Total cents is $1771 = 7 \cdot 11 \cdot 23.$ Writing (students)(pencils each)(cost per pencil) $= 1771,$ the number of students is a divisor of 1771 that is a majority of $30,$ hence more than $15.$ The only such divisor is $23.$

Then (pencils)(cost) $= 77 = 7 \cdot 11$ with cost $>$ pencils $> 1,$ forcing 7 pencils at 11 cents each.

Thus, the correct answer is **B**.

8. In the eight-term sequence $A, B, C, D, E, F, G, H,$ the value of C is 5 and the sum of any three consecutive terms is $30.$ What is $A + H?$

- A 17
- B 18
- C 25
- D 26
- E 43

Solution:

Since $A + B + C = B + C + D = 30,$ we get $D = A,$ and likewise the sequence repeats with period $3.$ Thus $H,$ the eighth term, equals $B.$

From $A + B + C = 30$ and $C = 5,$ we have $A + H = A + B = 30 - 5 = 25.$

Thus, the correct answer is **C**.

9. At a twins and triplets convention, there were 9 sets of twins and 6 sets of triplets, all from different families. Each twin shook hands with all the twins except his/her sibling and with half the triplets. Each triplet shook hands with all the triplets except his/her siblings and with half the twins. How many handshakes took place?

- A 324
- B 441
- C 630
- D 648
- E 882

Solution:

There are 18 twins and 18 triplets.

Twin-twin handshakes: each twin shakes $18 - 2 = 16$ other twins, giving $\frac{18 \cdot 16}{2} = 144.$

Triplet-triplet handshakes: each triplet shakes $18 - 3 = 15$ other triplets, giving $\frac{18 \cdot 15}{2} = 135.$

Twin-triplet handshakes: each twin shakes half the 18 triplets, giving $18 \cdot 9 = 162$ (each such handshake counted once).

The total is $144 + 135 + 162 = 441.$

Thus, the correct answer is **B**.

10. A pair of standard 6-sided fair dice is rolled once. The sum of the numbers rolled determines the diameter of a circle. What is the probability that the numerical value of the area of the circle is less than the numerical value of the circle's circumference?

A $\frac{1}{36}$

B $\frac{1}{12}$

C $\frac{1}{6}$

D $\frac{1}{4}$

E $\frac{5}{18}$

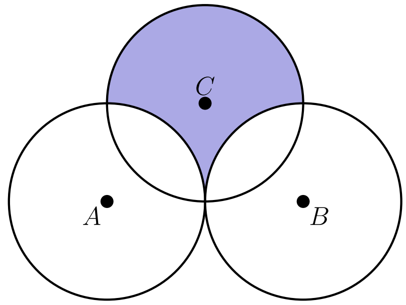
Solution:

For diameter d, area $<$ circumference means $\frac{\pi d^2}{4} < \pi d$, i.e. $d < 4$. Since $d \geq 2$, this needs a sum of 2 or 3.

A sum of 2 has probability $\frac{1}{36}$ and a sum of 3 has probability $\frac{2}{36}$, totaling $\frac{3}{36} = \frac{1}{12}$.

Thus, the correct answer is **B**.

11. Circles $A, B,$ and C each have radius $1.$ Circles A and B share one point of tangency. Circle C has a point of tangency with the midpoint of $\overline{AB}.$ What is the area inside circle C but outside circle A and circle B?



- A $3 - \frac{\pi}{2}$
- B $\frac{\pi}{2}$
- C 2
- D $\frac{3\pi}{4}$
- E $1 + \frac{\pi}{2}$

Solution:

Place $A = (-1, 0), B = (1, 0),$ so their tangency point is the origin, the midpoint of $\overline{AB}.$ Then $C = (0, 1),$ since C passes through the origin.

The distance from C to A (and to B) is $\sqrt{2}.$ Two unit circles whose centers are $\sqrt{2}$ apart overlap in a lens of area

$$2\cos^{-1}\left(\tfrac{\sqrt{2}}{2}\right) - \tfrac{\sqrt{2}}{2}\sqrt{4-2} = 2 \cdot \tfrac{\pi}{4} - 1 = \tfrac{\pi}{2} - 1.$$

Circles A and B meet only at the origin, so the two lenses do not overlap. The wanted area is

$$\pi - 2\left(\tfrac{\pi}{2} - 1\right) = 2.$$

Thus, the correct answer is **C**.

12. A power boat and a raft both left dock A on a river and headed downstream. The raft drifted at the speed of the river current. The power boat maintained a constant speed with respect to the river. The power boat reached dock B downriver, then immediately turned and traveled back upriver. It eventually met the raft on the river 9 hours after leaving dock A. How many hours did it take the power boat to go from A to B?

- A 3
- B 3.5
- C 4
- D 4.5
- E 5

Solution:

Measure everything relative to the water. In that frame the raft is stationary at the point where the boat started, and the boat moves at its constant speed v relative to the water, both downstream and upstream.

The boat leaves the raft, travels away for some time, then returns to it at the same relative speed, so it spends equal times going and returning. Hence the outbound leg to B takes half of 9, which is 4.5 hours.

Thus, the correct answer is **D**.

13. Triangle ABC has side-lengths $AB = 12, BC = 24$, and $AC = 18$. The line through the incenter of $\triangle ABC$ parallel to $\overline{BC}$ intersects $\overline{AB}$ at M and $\overline{AC}$ at N. What is the perimeter of $\triangle AMN$?

- A 27
- B 30
- C 33
- D 36
- E 42

Solution:

Let I be the incenter. Because $\overline{BI}$ bisects $\angle B$ and $MN \parallel BC$, alternate angles give $\angle MIB = \angle IBC = \angle MBI$, so $\triangle MBI$ is isosceles with $MB = MI$. Similarly $NC = NI$.

Therefore the perimeter of $\triangle AMN$ is

$$AM + MN + NA = AM + (MI + IN) + NA = AM + MB + NC + NA = AB + AC = 12 + 18 = 30.$$

Thus, the correct answer is **B**.

14. Suppose a and b are single-digit positive integers chosen independently and at random. What is the probability that the point (a, b) lies above the parabola $y = ax^2 - bx$?

A $\frac{11}{81}$

B $\frac{13}{81}$

C $\frac{5}{27}$

D $\frac{17}{81}$

E $\frac{19}{81}$

Solution:

Substituting $x = a, y = b$, the point is above the parabola when $b > a^3 - ab$, i.e. $b(a+1) > a^3$.

For $a = 1 : b > \frac{1}{2}$, all 9 values work. For $a = 2 : b > \frac{8}{3}$, so $b \geq 3$, giving 7. For $a = 3 : b > \frac{27}{4} = 6.75$, so $b \geq 7$, giving 3. For $a \geq 4$, no $b \leq 9$ works.

The count is $9 + 7 + 3 = 19$ out of 81, so the probability is $\frac{19}{81}$.

Thus, the correct answer is **E**.

15. The circular base of a hemisphere of radius 2 rests on the base of a square pyramid of height $6.$ The hemisphere is tangent to the other four faces of the pyramid. What is the edge-length of the base of the pyramid?

A $3\sqrt{2}$

B $\frac{13}{3}$

C $4\sqrt{2}$

D 6

E $\frac{13}{2}$

Solution:

Let the base have side $s,$ centered at the origin, with apex at height $6.$ Cut with the vertical plane through the apex and the midpoints of two opposite base edges. The slant face appears as the line from $\left(\frac{s}{2}, 0\right)$ to $(0, 6).$

This line is $\frac{2}{s}x + \frac{1}{6}y = 1.$ The hemisphere is tangent to the face, so the distance from the origin to this line is the radius $2:$

$$\frac{1}{\sqrt{\frac{4}{s^2} + \frac{1}{36}}} = 2.$$

Then $\frac{4}{s^2} + \frac{1}{36} = \frac{1}{4}$, so $\frac{4}{s^2} = \frac{2}{9}$ and $s^2 = 18,$ giving $s = 3\sqrt{2}.$

Thus, the correct answer is **A**.

16. Each vertex of convex pentagon $ABCDE$ is to be assigned a color. There are 6 colors to choose from, and the ends of each diagonal must have different colors. How many different colorings are possible?

- A 2520
- B 2880
- C 3120
- D 3250
- E 3750

Solution:

The diagonals connect the vertices in the order $A - C - E - B - D - A$, which is a 5-cycle. The condition is exactly that this cycle is properly colored.

The number of proper k-colorings of a cycle of length n is $(k-1)^n + (-1)^n(k-1)$. With $n = 5$ and $k = 6$,

$$5^5 + (-1)^5 \cdot 5 = 3125 - 5 = 3120.$$

Thus, the correct answer is **C**.

17. Circles with radii $1, 2,$ and 3 are mutually externally tangent. What is the area of the triangle determined by the points of tangency?

A $\frac{3}{5}$

B $\frac{4}{5}$

C 1

D $\frac{6}{5}$

E $\frac{4}{3}$

Solution:

The centers are separated by the sums of radii: $3, 4,$ and $5,$ a right triangle with the right angle at the radius-1 center. Place that center at $(0,0)$, the radius-2 center at $(3,0)$, and the radius-3 center at $(0,4)$.

The tangency points lie on the segments at distances equal to the radii: $(1,0), (0,1)$, and on the hypotenuse at $(3,0) + 2 \cdot \frac{(-3,4)}{5} = \left(\frac{9}{5}, \frac{8}{5}\right)$.

By the shoelace formula the area is

$$\tfrac{1}{2}\left|1\left(1-\tfrac{8}{5}\right)+0+\tfrac{9}{5}(0-1)\right| = \tfrac{1}{2}\cdot\tfrac{12}{5} = \tfrac{6}{5}.$$

Thus, the correct answer is **D**.

18. Suppose that $|x+y|+|x-y|=2$. What is the maximum possible value of x^2-6x+y^2?

- A 5
- B 6
- C 7
- D 8
- E 9

Solution:

The identity $|x+y|+|x-y|=2\max(|x|,|y|)$ turns the condition into $\max(|x|,|y|)=1$, the boundary of the square with $|x|\leq 1$ and $|y|\leq 1$.

On this region x^2-6x+y^2 increases as x decreases and as y^2 increases, so the maximum is at $x=-1, y=\pm 1:$

$$1+6+1=8.$$

Thus, the correct answer is **D**.

19. At a competition with N players, the number of players given elite status is equal to

$$2^{1+\lfloor \log_2(N-1)\rfloor}-N.$$

Suppose that 19 players are given elite status. What is the sum of the two smallest possible values of N?

Note: $\lfloor x\rfloor$ is the greatest integer less than or equal to x.

- A 38
- B 90
- C 154
- D 406
- E 1024

Solution:

Let $m=\lfloor \log_2(N-1)\rfloor$, so the elite count is $2^{m+1}-N=19$, giving $N=2^{m+1}-19$.

Consistency requires $2^m\leq N-1=2^{m+1}-20$, i.e. $2^m\geq 20$, so $m\geq 5$.

The two smallest choices are $m=5$ giving $N=64-19=45$, and $m=6$ giving $N=128-19=109$. Their sum is 154.

Thus, the correct answer is **C**.

20. Let $f(x) = ax^2 + bx + c$, where $a, b,$ and c are integers. Suppose that $f(1) = 0, 50 < f(7) < 60, 70 < f(8) < 80$, and $5000k < f(100) < 5000(k+1)$ for some integer k. What is k?

A 1

B 2

C 3

D 4

E 5

Solution:

Since $f(1) = a + b + c = 0$, we have $c = -a - b$. Then

$$f(7) = 48a + 6b = 6(8a + b), \qquad f(8) = 63a + 7b = 7(9a + b).$$

From $50 < 6(8a + b) < 60$ we get $8a + b = 9$, and from $70 < 7(9a + b) < 80$ we get $9a + b = 11$. Subtracting, $a = 2$, then $b = -7$ and $c = 5$.

So $f(100) = 20000 - 700 + 5 = 19305$, which lies in $5000 \cdot 3 < 19305 < 5000 \cdot 4$, giving $k = 3$.

Thus, the correct answer is **C**.

21. Let $f_1(x) = \sqrt{1 - x}$, and for integers $n \geq 2$, let $f_n(x) = f_{n-1}\left(\sqrt{n^2 - x}\right)$. If N is the largest value of n for which the domain of f_n is nonempty, the domain of f_N is $\{c\}$. What is $N + c$?

A -226

B -144

C -20

D 20

E 144

Solution:

Each step requires $\sqrt{n^2 - x}$ to lie in the domain of f_{n-1}. Tracking the domains:

$f_1 : (-\infty, 1]$. $f_2 : \sqrt{4 - x} \in (-\infty, 1] \Rightarrow [3, 4]$. $f_3 : \sqrt{9 - x} \in [3, 4] \Rightarrow [-7, 0]$. $f_4 : \sqrt{16 - x} \in [-7, 0] \Rightarrow \{16\}$ (only the value 0 is possible). $f_5 : \sqrt{25 - x} = 16 \Rightarrow \{-231\}$.

For f_6 we would need $\sqrt{36 - x} = -231$, impossible, so the domain is empty. Hence $N = 5, c = -231$, and $N + c = -226$.

Thus, the correct answer is **A**.

22. Let R be a square region and $n \geq 4$ an integer. A point X in the interior of R is called n-ray partitional if there are n rays emanating from X that divide R into n triangles of equal area. How many points are 100-ray partitional but not 60-ray partitional?

- A 1500
- B 1560
- C 2320
- D 2480
- E 2500

Solution:

For even $n = 2m$, the n-ray partitional points are exactly $\left(\frac{i}{m}, \frac{j}{m}\right)$ with $1 \leq i, j \leq m - 1$, giving $(m-1)^2$ points.

For $n = 100$ ($m = 50$) there are $49^2 = 2401$ points. A point is both 100- and 60-ray partitional iff its coordinates are multiples of $\frac{1}{10}$, i.e. it is 20-ray partitional, giving $9^2 = 81$ points.

So the count is $2401 - 81 = 2320.$

Thus, the correct answer is **C**.

23. Let $f(z) = \dfrac{z+a}{z+b}$ and $g(z) = f(f(z))$, where a and b are complex numbers. Suppose that $|a| = 1$ and $g(g(z)) = z$ for all z for which $g(g(z))$ is defined. What is the difference between the largest and smallest possible values of $|b|$?

- A 0
- B $\sqrt{2} - 1$
- C $\sqrt{3} - 1$
- D 1
- E 2

Solution:

Represent f by $M = \begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix}$. Then $g(g(z)) = z$ says f composed with itself four times is the identity, so M^4 is a scalar matrix.

This happens when the ratio of eigenvalues is a fourth root of unity. The order-4 case gives $(\operatorname{tr} M)^2 = 2 \det M$, i.e. $(1+b)^2 = 2(b-a)$, which simplifies to $b^2 = -(1+2a)$. The order-2 case gives $b = -1.$

Then $|b|^2 = |1 + 2a|$, and as a runs over $|a| = 1$, $|1 + 2a|$ ranges over $[1, 3]$, so $|b|$ ranges over $[1, \sqrt{3}]$ (the value $b = -1$ is included). The difference is $\sqrt{3} - 1.$

Thus, the correct answer is **C**.

24. Consider all quadrilaterals $ABCD$ such that $AB = 14, BC = 9, CD = 7,$ and $DA = 12.$ What is the radius of the largest possible circle that fits inside or on the boundary of such a quadrilateral?

- A $\sqrt{15}$
- B $\sqrt{21}$
- C $2\sqrt{6}$
- D 5
- E $2\sqrt{7}$

Solution:

Because $AB + CD = 14 + 7 = 21 = 9 + 12 = BC + DA,$ a tangential quadrilateral (one with an inscribed circle) with these sides exists. For a tangential quadrilateral the area equals $r \cdot s$ with semiperimeter $s = 21,$ so maximizing r means maximizing the area.

Among tangential quadrilaterals with given sides, the largest area is achieved by the cyclic (bicentric) one, whose area is

$$\sqrt{(s-a)(s-b)(s-c)(s-d)} = \sqrt{7 \cdot 12 \cdot 14 \cdot 9} = 42\sqrt{6}.$$

Then $r = \dfrac{42\sqrt{6}}{21} = 2\sqrt{6}.$

Thus, the correct answer is **C**.

25. Triangle ABC has $\angle BAC = 60°$, $\angle CBA \leq 90°$, $BC = 1$, and $AC \geq AB$. Let H, I, and O be the orthocenter, incenter, and circumcenter of $\triangle ABC$, respectively. Assume that the area of the pentagon $BCOIH$ is the maximum possible. What is $\angle CBA$?

A $60°$

B $72°$

C $75°$

D $80°$

E $90°$

Solution:

When $\angle BAC = 60°$, a classical fact is that B, C, O, I, and H all lie on a common circle, so $BCOIH$ is a convex cyclic pentagon whose vertices depend only on the shape of the triangle.

Fixing $BC = 1$ and $\angle A = 60°$, the circumradius is $R = \frac{1}{\sqrt{3}}$, and O, I, H are determined by $\angle CBA = B$ (with $\angle BCA = 120° - B$). Writing the pentagon area as a function of B on the allowed range $60° \leq B \leq 90°$ and maximizing gives an interior maximum at $B = 80°$.

So the maximizing angle is $\angle CBA = 80°$.

Thus, the correct answer is **D**.

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