2010 AMC 12B Solutions

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1. Makayla attended two meetings during her 9-hour work day. The first meeting took 45 minutes and the second meeting took twice as long. What percent of her work day was spent attending meetings?

A 15

B 20

C 25

D 30

E 35

Solution:

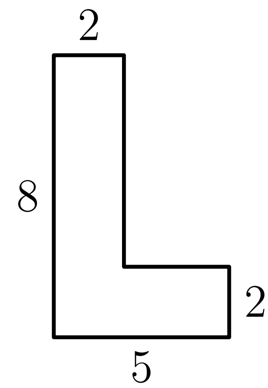
The two meetings lasted $45 + 90 = 135$ minutes, and the work day is $9 \cdot 60 = 540$ minutes.

The fraction of the day spent in meetings is

$$\frac{135}{540} = \frac{1}{4} = 25\%.$$

Thus, the correct answer is **C**.

2. A big L is formed as shown. What is its area?



A 22

B 24

C 26

D 28

E 30

Solution:

The region splits into an 8×2 vertical rectangle and a 2×3 horizontal foot, whose width is $5 - 2 = 3$.

The total area is

$$8 \cdot 2 + 3 \cdot 2 = 16 + 6 = 22.$$

Thus, the correct answer is **A**.

3. A ticket to a school play costs x dollars, where x is a whole number. A group of 9th graders buys tickets costing a total of $\$48$, and a group of 10th graders buys tickets costing a total of $\$64$. How many values for x are possible?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

The price x must divide both totals, so x is a common divisor of 48 and 64.

Since $\gcd(48, 64) = 16$, the common divisors are $1, 2, 4, 8,$ and 16. There are 5 possible values.

Thus, the correct answer is **E**.

4. A month with 31 days has the same number of Mondays and Wednesdays. How many of the seven days of the week could be the first day of this month?

A 2

B 3

C 4

D 5

E 6

Solution:

Since $31 = 4 \cdot 7 + 3,$ the first three days of the month each occur five times, and the other four days occur four times.

Mondays and Wednesdays are equal in number exactly when both fall in the five-time group or both fall in the four-time group.

If the first day is Monday, the five-time days are Mon, Tue, Wed (both appear five times). If the first day is Thursday or Friday, the five-time days miss both Monday and Wednesday (both appear four times). Every other starting day includes exactly one of Monday or Wednesday.

So the first day can be Monday, Thursday, or Friday, giving 3 possibilities.

Thus, the correct answer is **B**.

5. Lucky Larry's teacher asked him to substitute numbers for $a, b, c, d,$ and e in the expression

$$a - (b - (c - (d + e)))$$

and evaluate the result. Larry ignored the parentheses but added and subtracted correctly and obtained the correct result by coincidence. The numbers Larry substituted for $a, b, c,$ and d were $1, 2, 3,$ and $4,$ respectively. What number did Larry substitute for $e?$

A -5

B -3

C 0

D 3

E 5

Solution:

The correct value is $a - (b - (c - (d + e))) = a - b + c - d - e.$ With $a, b, c, d = 1, 2, 3, 4,$ this equals $1 - 2 + 3 - 4 - e = -2 - e.$

Larry dropped the parentheses and computed $1 - 2 - 3 - 4 + e = -8 + e.$

Setting $-2 - e = -8 + e$ gives $2e = 6,$ so $e = 3.$

Thus, the correct answer is **D**.

6. At the beginning of the school year, 50% of all students in Mr. Wells' math class answered "Yes" to the question "Do you love math", and 50% answered "No." At the end of the school year, 70% answered "Yes" and 30% answered "No." Altogether, $x\%$ of the students gave a different answer at the beginning and end of the school year. What is the difference between the maximum and the minimum possible values of x?

A 0

B 20

C 40

D 60

E 80

Solution:

Assume 100 students. The number of "Yes" answers rises from 50 to $70,$ so at least $70 - 50 = 20$ students switched from "No" to "Yes"; thus $x \geq 20.$

Since only 30 students answer "No" at the end, at least $50 - 30 = 20$ of the original "Yes" students still answer "Yes," so at most 80 students switched; thus $x \leq 80.$

Both extremes are achievable, so the difference is $80 - 20 = 60.$

Thus, the correct answer is **D**.

7. Shelby drives her scooter at a speed of 30 miles per hour if it is not raining, and 20 miles per hour if it is raining. Today she drove in the sun in the morning and in the rain in the evening, for a total of 16 miles in 40 minutes. How many minutes did she drive in the rain?

- A 18
- B 21
- C 24
- D 27
- E 30

Solution:

Let t be the number of minutes driven in the rain. She covers $20 \cdot \frac{t}{60}$ miles in the rain and $30 \cdot \frac{40-t}{60}$ miles in the sun.

Setting the total to 16 gives

$$\frac{20t + 30(40-t)}{60} = 16,$$

so $1200 - 10t = 960$ and $t = 24.$

Thus, the correct answer is **C**.

8. Every high school in the city of Euclid sent a team of 3 students to a math contest. Each participant in the contest received a different score. Andrea's score was the median among all students, and hers was the highest score on her team. Andrea's teammates Beth and Carla placed 37th and 64th, respectively. How many schools are in the city?

- A 22
- B 23
- C 24
- D 25
- E 26

Solution:

With n schools there are $3n$ students. Carla placed 64th, so $3n \geq 64$ and $n \geq 22.$

The scores are distinct and Andrea is the median, so $3n$ is odd, forcing n odd and $n \geq 23.$

Andrea's position is $\dfrac{3n+1}{2}$, and she beat Beth (37th), so $\dfrac{3n+1}{2} < 37$, giving $3n < 73$ and $n \leq 24.$ The only odd value is $n = 23.$

Thus, the correct answer is **B**.

9. Let n be the smallest positive integer such that n is divisible by 20, n^2 is a perfect cube, and n^3 is a perfect square. What is the number of digits of n?

- A 3
- B 4
- C 5
- D 6
- E 7

Solution:

To be smallest, n uses only the primes of 20, so $n = 2^a \cdot 5^b$ with $a \geq 2$ and $b \geq 1$.

Since $n^2 = 2^{2a}5^{2b}$ is a perfect cube, $3 \mid a$ and $3 \mid b$. Since $n^3 = 2^{3a}5^{3b}$ is a perfect square, $2 \mid a$ and $2 \mid b$. Hence $6 \mid a$ and $6 \mid b$.

The smallest choice is $a = b = 6$, so $n = 2^6 \cdot 5^6 = 10^6 = 1{,}000{,}000$, which has 7 digits.

Thus, the correct answer is **E**.

10. The average of the numbers $1, 2, 3, \ldots, 98, 99,$ and x is $100x$. What is x?

A $\frac{49}{101}$

B $\frac{50}{101}$

C $\frac{1}{2}$

D $\frac{51}{101}$

E $\frac{50}{99}$

Solution:

The numbers 1 through 99 sum to $\frac{99 \cdot 100}{2} = 4950.$

The average condition is

$$\frac{4950 + x}{100} = 100x,$$

so $4950 + x = 10000x$ and $9999x = 4950.$

Thus $x = \frac{4950}{9999} = \frac{50}{101}.$

Thus, the correct answer is **B**.

11. A palindrome between 1000 and $10{,}000$ is chosen at random. What is the probability that it is divisible by 7?

A $\frac{1}{10}$

B $\frac{1}{9}$

C $\frac{1}{7}$

D $\frac{1}{6}$

E $\frac{1}{5}$

Solution:

A four-digit palindrome has the form $\overline{abba} = 1001a + 110b$ with $1 \leq a \leq 9$ and $0 \leq b \leq 9$.

Since $1001 = 7 \cdot 11 \cdot 13$ is divisible by 7 and 110 is not, the number is divisible by 7 exactly when $7 \mid b$, that is $b = 0$ or $b = 7$.

For each a, that is 2 of the 10 choices of b, a probability of $\frac{2}{10} = \frac{1}{5}$.

Thus, the correct answer is **E**.

12. For what value of x does

$$\log_{\sqrt{2}} \sqrt{x} + \log_2 x + \log_4 \left(x^2\right) + \log_8 \left(x^3\right) + \log_{16} \left(x^4\right) = 40?$$

- A 8
- B 16
- C 32
- D 256
- E 1024

Solution:

Let $L = \log_2 x$. Converting each term to base 2 :

$\log_{\sqrt{2}} \sqrt{x} = \dfrac{L/2}{1/2} = L, \log_2 x = L, \log_4 x^2 = \dfrac{2L}{2} = L, \log_8 x^3 = \dfrac{3L}{3} = L$, and $\log_{16} x^4 = \dfrac{4L}{4} = L.$

The equation becomes $5L = 40$, so $L = 8$ and $x = 2^8 = 256.$

Thus, the correct answer is **D**.

13. In $\triangle ABC, \cos(2A - B) + \sin(A + B) = 2$ and $AB = 4.$ What is BC?

A $\sqrt{2}$

B $\sqrt{3}$

C 2

D $2\sqrt{2}$

E $2\sqrt{3}$

Solution:

A cosine plus a sine equals 2 only when each equals $1.$ So $\cos(2A - B) = 1$ and $\sin(A + B) = 1,$ giving $2A - B = 0°$ and $A + B = 90°.$

Solving, $A = 30°$ and $B = 60°,$ so $\triangle ABC$ is a $30\text{-}60\text{-}90$ right triangle with the right angle at $C.$

With hypotenuse $AB = 4,$ the side BC opposite the $30°$ angle is half the hypotenuse, so $BC = 2.$

Thus, the correct answer is **C**.

14. Let $a, b, c, d,$ and e be positive integers with $a + b + c + d + e = 2010,$ and let M be the largest of the sums $a + b, b + c, c + d,$ and $d + e.$ What is the smallest possible value of M?

A 670

B 671

C 802

D 803

E 804

Solution:

Each of $a + b, d + e,$ and c is at most M (note $c \leq c + d \leq M$). Adding, $2010 = (a + b) + c + (d + e) \leq 3M,$ so $M \geq 670.$

If $M = 670,$ then $c = 670,$ but then $b + c \geq 671 > M,$ a contradiction. Hence $M \geq 671.$

The value 671 is reached by $(a, b, c, d, e) = (669, 1, 670, 1, 669),$ whose consecutive-pair sums are $670, 671, 671, 670.$

Thus, the correct answer is **B**.

15. For how many ordered triples (x, y, z) of nonnegative integers less than 20 are there exactly two distinct elements in the set $\{i^x, (1+i)^y, z\}$, where $i = \sqrt{-1}$?

A 149

B 205

C 215

D 225

E 235

Solution:

We need exactly two of $i^x, (1+i)^y, z$ equal, with the third different. The three cases are the three possible equal pairs.

Case $i^x = (1+i)^y$: since $|i^x| = 1$ but $|(1+i)^y| = 2^{y/2} > 1$ for $y \geq 1,$ we need $y = 0,$ so $(1+i)^0 = 1$ and $i^x = 1,$ i.e. $x \in \{0, 4, 8, 12, 16\}.$ Then z is any of the 19 values other than $1.$ This gives $5 \cdot 19 = 95$ triples.

Case $i^x = z$: the only nonnegative-integer value of i^x is 1 (with x a multiple of 4), so $z = 1$ and $(1+i)^y \neq 1,$ meaning $y \geq 1.$ This gives $5 \cdot 19 = 95$ triples.

Case $(1+i)^y = z$: since $(1+i)^2 = 2i,$ the power $(1+i)^y$ is a nonnegative integer below 20 only for $y = 0$ (value 1) or $y = 8$ (value 16). If $y = 0,\ z = 1,$ we need $i^x \neq 1,$ so x is not a multiple of 4 (15 values). If $y = 8,\ z = 16,$ then i^x is never $16,$ so x is free (20 values). This gives $15 + 20 = 35$ triples.

Altogether $95 + 95 + 35 = 225.$

Thus, the correct answer is **D**.

16. Positive integers $a, b,$ and c are randomly and independently selected with replacement from the set $\{1, 2, 3, \ldots, 2010\}$. What is the probability that $abc + ab + a$ is divisible by 3?

A $\frac{1}{3}$

B $\frac{29}{81}$

C $\frac{31}{81}$

D $\frac{11}{27}$

E $\frac{13}{27}$

Solution:

Factor $abc + ab + a = a(bc + b + 1)$. Since 2010 is a multiple of 3, each of a, b, c is uniform modulo 3.

If $3 \mid a$ (probability $\frac{1}{3}$), the product is divisible by 3.

If $3 \nmid a$ (probability $\frac{2}{3}$), we need $3 \mid bc + b + 1$. Checking residues, this holds exactly when $(b, c) \equiv (1, 1)$ or $(2, 0) \pmod 3$, a probability of $\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9}$.

The total probability is

$$\frac{1}{3} + \frac{2}{3} \cdot \frac{2}{9} = \frac{1}{3} + \frac{4}{27} = \frac{13}{27}.$$

Thus, the correct answer is **E**.

17. The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?

- A 18
- B 24
- C 36
- D 42
- E 60

Solution:

Write a_{ij} for the entry in row i, column j. The conditions force $a_{11} = 1, a_{33} = 9$, and $a_{22} \in \{4, 5, 6\}$.

If $a_{22} = 4$, then $\{a_{12}, a_{21}\} = \{2, 3\}$ and $\{5, 6, 7, 8\}$ split as complementary pairs filling the rest of the last row and column: $\binom{4}{2} = 6$ splits times 2 orders for $\{2, 3\}$ gives 12 arrays. By symmetry $a_{22} = 6$ also gives 12.

If $a_{22} = 5$, then $\{a_{12}, a_{13}, a_{23}\}$ and $\{a_{21}, a_{31}, a_{32}\}$ are complementary subsets of $\{2, 3, 4, 6, 7, 8\}$ subject to the ordering constraints, forcing the first set to be $\{2, 3, 4\}$ or $\{6, 7, 8\}$; this gives $\binom{6}{3} - 2 = 18$ arrays.

Altogether $12 + 12 + 18 = 42$.

Thus, the correct answer is **D**.

18. A frog makes 3 jumps, each exactly 1 meter long. The directions of the jumps are chosen independently and at random. What is the probability that the frog's final position is no more than 1 meter from its starting position?

A $\frac{1}{6}$

B $\frac{1}{5}$

C $\frac{1}{4}$

D $\frac{1}{3}$

E $\frac{1}{2}$

Solution:

This is a continuous (geometric) probability. Anchor the second jump from $P = (0, 0)$ to $Q = (1, 0)$, and let α, β be the directions of the first and third jumps, so the start is $A = (\cos\alpha, \sin\alpha)$ and the end is $B = (1 + \cos\beta, \sin\beta)$.

Taking $0 \leq \alpha \leq \pi$ and $0 \leq \beta \leq 2\pi$, the requirement $AB \leq 1$ holds exactly when $\alpha \leq \beta \leq \pi$.

In the $\alpha\beta$-rectangle of area $2\pi^2$, the favorable region is a triangle of area $\frac{\pi^2}{2}$, so the probability is $\dfrac{\pi^2/2}{2\pi^2} = \dfrac{1}{4}$.

Thus, the correct answer is **C**.

19. A high school basketball game between the Raiders and the Wildcats was tied at the end of the first quarter. The number of points scored by the Raiders in each of the four quarters formed an increasing geometric sequence, and the number of points scored by the Wildcats in each of the four quarters formed an increasing arithmetic sequence. At the end of the fourth quarter, the Raiders had won by one point. Neither team scored more than 100 points. What was the total number of points scored by the two teams in the first half?

- A 30
- B 31
- C 32
- D 33
- E 34

Solution:

Let the Raiders score a, ar, ar^2, ar^3 (increasing geometric, $r > 1$) and the Wildcats $a, a+d, a+2d, a+3d$ (increasing arithmetic), tied in the first quarter at a.

Every quarter score is a positive integer and each total is under 100, so the ratio and first term are small. Testing $r = 2$ gives Raiders $5, 10, 20, 40$ with total 75.

The Wildcats then total $74 = 4a + 6d = 20 + 6d$, so $d = 9$, giving $5, 14, 23, 32$. The Raiders won 75 to 74.

The first-half total is $(5 + 10) + (5 + 14) = 34$.

Thus, the correct answer is **E**.

20. A geometric sequence (a_n) has $a_1 = \sin x, a_2 = \cos x$, and $a_3 = \tan x$ for some real number x. For what value of n does $a_n = 1 + \cos x$?

A 4

B 5

C 6

D 7

E 8

Solution:

The common ratio is $r = \dfrac{a_2}{a_1} = \cot x$. Then $a_4 = a_3 \cdot r = \tan x \cot x = 1$.

From $a_3 = a_1 r^2$, we get $\tan x = \sin x \cot^2 x = \dfrac{\cos^2 x}{\sin x}$, so $\sin^2 x = \cos^3 x$, i.e. $(\cos^2 x)(1 + \cos x) = 1$.

Hence $1 + \cos x = \dfrac{1}{\cos^2 x}$. Also $r^2 = \dfrac{\cos^2 x}{\sin^2 x} = \dfrac{\cos^2 x}{\cos^3 x} = \dfrac{1}{\cos x}$, so $r^4 = \dfrac{1}{\cos^2 x} = 1 + \cos x$.

Therefore $1 + \cos x = a_4 \cdot r^4 = a_8$, so $n = 8$.

Thus, the correct answer is **E**.

21. Let $a > 0$, and let $P(x)$ be a polynomial with integer coefficients such that

$$P(1) = P(3) = P(5) = P(7) = a,$$

and

$$P(2) = P(4) = P(6) = P(8) = -a.$$

What is the smallest possible value of a?

A 105

B 315

C 945

D $7!$

E $8!$

Solution:

Since $1, 3, 5, 7$ are roots of $P(x) - a$, write $P(x) - a = (x-1)(x-3)(x-5)(x-7)Q(x)$ with Q having integer coefficients.

Evaluating at $x = 2, 4, 6, 8$ (where $P = -a$) gives

$$-2a = -15\,Q(2) = 9\,Q(4) = -15\,Q(6) = 105\,Q(8).$$

So $15, 9$, and 105 all divide $2a$, hence $\mathrm{lcm}(15, 9, 105) = 315$ divides $2a$. Since 315 is odd, $315 \mid a$, so $a \geq 315$.

The value $a = 315$ is attainable, so the smallest is 315.

Thus, the correct answer is **B**.

22. Let $ABCD$ be a cyclic quadrilateral. The side lengths of $ABCD$ are distinct integers less than 15 such that $BC \cdot CD = AB \cdot DA$. What is the largest possible value of BD?

A $\sqrt{\frac{325}{2}}$

B $\sqrt{185}$

C $\sqrt{\frac{389}{2}}$

D $\sqrt{\frac{425}{2}}$

E $\sqrt{\frac{533}{2}}$

Solution:

Let $a = AB, b = BC, c = CD, d = DA$ and $k = bc = ad$. Writing each triangle's area in terms of the circumradius and using $[ABC] + [CDA] = [BCD] + [ABD]$ gives $(ab + cd) \cdot AC = 2k \cdot BD$.

Ptolemy's theorem gives $AC \cdot BD = ac + bd$. Eliminating AC,

$$BD^2 = \frac{(ac + bd)(ab + cd)}{2k} = \frac{1}{2}\left(a^2 + b^2 + c^2 + d^2\right).$$

The sides are distinct integers below 15 with $bc = ad$, so neither 11 nor 13 can appear (each is prime and would need a matching factor on the other side).

To maximize, take the largest side 14. Writing the others as $s_1 > s_2 > s_3$ with $14s_3 = s_1 s_2$, the best case is $s_2 = 7$, giving $s_1 = 2s_3$ and $(a, b, c, d) = (14, 12, 7, 6)$. Then

$$2BD^2 = 14^2 + 12^2 + 7^2 + 6^2 = 425,$$

so $BD = \sqrt{\frac{425}{2}}$.

Thus, the correct answer is **D**.

23. Monic quadratic polynomials $P(x)$ and $Q(x)$ have the property that $P(Q(x))$ has zeros at $x = -23, -21, -17,$ and -15, and $Q(P(x))$ has zeros at $x = -59, -57, -51,$ and -49. What is the sum of the minimum values of $P(x)$ and $Q(x)$?

A -100

B -82

C -73

D -64

E 0

Solution:

Write $P(x) = (x - h_1)^2 - k_1^2$ and $Q(x) = (x - h_2)^2 - k_2^2$, with minimum values $-k_1^2$ and $-k_2^2$.

The zeros of $P(Q(x))$ occur where $Q(x) = h_1 \pm k_1$; their four solutions are symmetric about h_2, so h_2 is the average $\frac{-23-21-17-15}{4} = -19$. Then $Q(-15) - Q(-17) = (16 - k_2^2) - (4 - k_2^2) = 12$, and this difference equals $2k_1$, so $k_1 = 6$.

Symmetrically, $h_1 = \frac{-59-57-51-49}{4} = -54$, and $P(-49) - P(-51) = (25 - k_1^2) - (9 - k_1^2) = 16 = 2k_2$, so $k_2 = 8$.

The sum of the minimum values is $-k_1^2 - k_2^2 = -36 - 64 = -100$.

Thus, the correct answer is **A**.

24. The set of real numbers x for which

$$\frac{1}{x-2009}+\frac{1}{x-2010}+\frac{1}{x-2011}\geq 1$$

is the union of intervals of the form $a<x\leq b$. What is the sum of the lengths of these intervals?

A $\frac{1003}{335}$

B $\frac{1004}{335}$

C 3

D $\frac{403}{134}$

E $\frac{202}{67}$

Solution:

Let $f(x)$ be the left-hand side. On each interval between consecutive asymptotes $2009, 2010, 2011$, the function f is decreasing, and $f<1$ for all $x<2009$.

On each of $(2009, 2010)$, $(2010, 2011)$, and $(2011, \infty)$, the solution is the part from the left asymptote up to a value x_i where $f(x_i)=1$. So the solution set consists of three intervals with left endpoints $2009, 2010, 2011$ and right endpoints x_1, x_2, x_3.

The total length is $(x_1-2009)+(x_2-2010)+(x_3-2011)=x_1+x_2+x_3-6030$.

Clearing denominators in $f(x)=1$ gives

$$x^3-(2009+2010+2011+3)x^2+\cdots=0,$$

whose roots are x_1, x_2, x_3. By Vieta, $x_1+x_2+x_3=6033$, so the sum of lengths is $6033-6030=3$.

Thus, the correct answer is **C**.

25. For every integer $n \geq 2$, let $\text{pow}(n)$ be the largest power of the largest prime that divides n. For example, $\text{pow}(144) = \text{pow}(2^4 \cdot 3^2) = 3^2$. What is the largest integer m such that 2010^m divides

$$\prod_{n=2}^{5300} \text{pow}(n)?$$

A 74

B 75

C 76

D 77

E 78

Solution:

Since $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, write the product as $2^A 3^B 5^C 67^D$ times a factor coprime to all four primes; then $m = \min(A, B, C, D)$.

Prime 2 : $\text{pow}(n)$ is a power of 2 only when $n = 2^k$. Since $2^{12} = 4096 < 5300 < 2^{13}$, the values $k = 1, \ldots, 12$ contribute $A = 1 + 2 + \cdots + 12 = 78$.

Prime 67 : $\text{pow}(n) = 67$ when 67 is the largest prime factor, i.e. $n = 67j$ with $1 \leq j \leq 79$ and every prime factor of j at most 67; excluding $j = 67, 71, 73, 79$ leaves 75 values. The one n with $\text{pow}(n) = 67^2$ is $n = 67^2 < 5300$, adding 2. So $D = 75 + 2 = 77$.

A similar count shows the exponents of 3 and 5 are each at least 77.

Therefore $m = \min(78, B, C, 77) = 77$.

Thus, the correct answer is **D**.

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