2010 AMC 12A Solutions

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1. What is $(20-(2010-201))+(2010-(201-20))$?

- A -4020
- B 0
- C 40
- D 401
- E 4020

Solution:

Distributing the negative signs gives

$$(20-2010+201)+(2010-201+20)=40.$$

Thus, **C** is the correct answer.

2. A ferry boat shuttles tourists to an island every hour starting at 10 am until its last trip, which starts at 3 pm. One day the boat captain notes that on the 10 am trip there were 100 tourists on the ferry boat, and that on each successive trip, the number of tourists was 1 fewer than on the previous trip. How many tourists did the ferry take to the island that day?

- A 585
- B 594
- C 672
- D 679
- E 694

Solution:

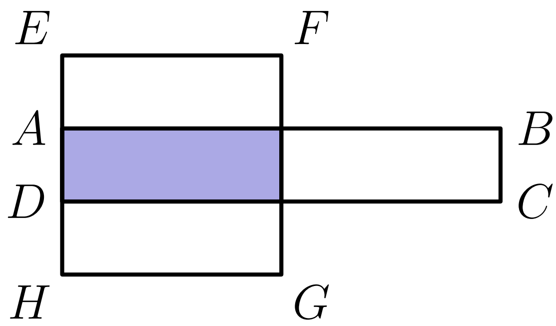
The ferry makes 6 trips: at $10, 11, 12, 1, 2,$ and $3.$

The numbers of tourists are $100, 99, 98, 97, 96, 95,$ so the total is

$$6 \cdot 100 - (1 + 2 + 3 + 4 + 5) = 600 - 15 = 585.$$

Thus, **A** is the correct answer.

3. Rectangle $ABCD$, pictured below, shares 50% of its area with square $EFGH$. Square $EFGH$ shares 20% of its area with rectangle $ABCD$. What is $\dfrac{AB}{AD}$?



A 4

B 5

C 6

D 8

E 10

Solution:

Let s be the side length of square $EFGH$. The shaded overlap has width s and height AD, so its area is $s \cdot AD$.

Because the overlap is 50% of the rectangle, $s \cdot AD = \frac{1}{2}\, AB \cdot AD$, so $AB = 2s$. Because it is 20% of the square, $s \cdot AD = \frac{1}{5}s^2$, so $AD = \frac{s}{5}$.

Therefore

$$\frac{AB}{AD} = \frac{2s}{s/5} = 10.$$

Thus, **E** is the correct answer.

4. If $x < 0$, then which of the following must be positive?

A $\frac{x}{|x|}$

B $-x^2$

C -2^x

D $-x^{-1}$

E $\sqrt[3]{x}$

Solution:

Choice (D) is $-x^{-1} = -\frac{1}{x}$. When $x < 0$, $\frac{1}{x} < 0$, so $-\frac{1}{x} > 0$.

Testing $x = -1$ shows the other choices need not be positive: $\frac{x}{|x|} = -1, -x^2 = -1,$ $-2^x = -\frac{1}{2}$, and $\sqrt[3]{x} = -1$.

Thus, **D** is the correct answer.

5. Halfway through a 100-shot archery tournament, Chelsea leads by 50 points. For each shot a bullseye scores 10 points, with other possible scores being $8, 4, 2,$ and 0 points. Chelsea always scores at least 4 points on each shot. If Chelsea's next n shots are bullseyes she will be guaranteed victory. What is the minimum value for n?

- A 38
- B 40
- C 42
- D 44
- E 46

Solution:

The opponent can score at most $50 \cdot 10 = 500$ on the last 50 shots. Since Chelsea leads by 50, she must score more than $500 - 50 = 450$ points on her remaining shots to guarantee victory.

Her n bullseyes give $10n$ points, and her other $50 - n$ shots give at least $4(50 - n)$ points, so

$$10n + 4(50 - n) > 450.$$

This simplifies to $6n > 250$, i.e. $n > 41\frac{2}{3}$.

Therefore Chelsea needs at least 42 bullseyes.

Thus, **C** is the correct answer.

6. A *palindrome*, such as $83438,$ is a number that remains the same when its digits are reversed. The numbers x and $x+32$ are three-digit and four-digit palindromes, respectively. What is the sum of the digits of $x?$

A 20

B 21

C 22

D 23

E 24

Solution:

Note that x is at most $999.$ This means that $x+32$ has a maximum of $1031.$

Similarly, we have that the minimum value of $x+32$ is $1000.$

The only palindrome in this range is $1001,$ so this is what $x+32$ equals.

Then

$$x+32=1001$$

$$x=969.$$

The sum of the digits is then

$$9+6+9=24.$$

Thus, **E** is the correct answer.

7. Logan is constructing a scaled model of his town. The city's water tower stands 40 meters high, and the top portion is a sphere that holds $100,000$ liters of water. Logan's miniature water tower holds 0.1 liters. How tall, in meters, should Logan make his tower?

A 0.04

B $\frac{0.4}{\pi}$

C 0.4

D $\frac{4}{\pi}$

E 4

Solution:

The miniature tower holds

$$\frac{100,000}{.1} = 1,000,000$$

times less water than the actual tower. Since this is the ratio for volumes, the ratio of heights is

$$(1,000,000)^{1/3} = 100.$$

This means that the height of the miniature tower is

$$\frac{40}{100} = .4.$$

Thus, **C** is the correct answer.

8. Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD, and suppose that $\triangle CFE$ is equilateral. What is $\angle ACB$?

A 60°

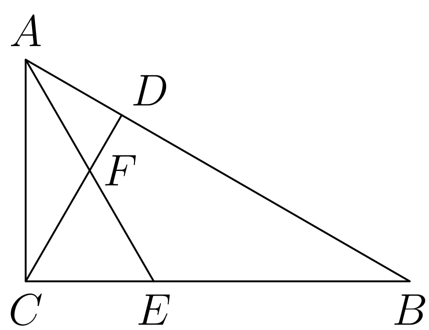
B 75°

C 90°

D 105°

E 120°

Solution:



Let $\angle BAE = \angle ACD = x$. Note that $\angle CFE = 60^\circ$ since $\triangle CFE$ is equilateral.

We then have that

$$\angle AFC = 180^\circ - \angle CFE = 120^\circ.$$

Then:

$$\begin{aligned}\angle FAC &= 180^\circ - 120^\circ - x \\ &= 60^\circ - x \\ &= \angle EAC.\end{aligned}$$

We then get that

$$\begin{aligned}\angle BAC &= \angle BAE + \angle EAC \\ &= x + 60^\circ - x \\ &= 60^\circ.\end{aligned}$$

Since $AB = 2 \cdot AC$ and $\angle BAC = 60^\circ$, we have that $\triangle ABC$ is a $30 - 60 - 90$ triangle.

Thus, **C** is the correct answer.

9. A solid cube has side length 3 inches. A 2-inch by 2-inch square hole is cut into the center of each face. The edges of each cut are parallel to the edges of the cube, and each hole goes all the way through the cube. What is the volume, in cubic inches, of the remaining solid?

- A 7
- B 8
- C 10
- D 12
- E 15

Solution:

Note that all the cut out solids intersect in the middle of the cube.

This region of intersection is a cube with side length $2.$ Then the volume of the cutout region is

$$\begin{aligned} 3 \cdot 2 \cdot 2 \cdot 3 - 2 \cdot 2^3 &= 36 - 16 \\ &= 20. \end{aligned}$$

We have to subtract out the center region twice since it is included in all 3 regions.

The remaining volume is then

$$3^3 - 20 = 27 - 20 = 7.$$

Thus, **A** is the correct answer.

10. The first four terms of an arithmetic sequence are $p, 9, 3p-q$, and $3p+q$. What is the 2010th term of this sequence?

- A 8041
- B 8043
- C 8045
- D 8047
- E 8049

Solution:

Consecutive terms differ by a common difference d. From the last two terms, $d = (3p+q)-(3p-q) = 2q$.

From the first two terms, $9 - p = d = 2q$, and from the second and third, $(3p-q) - 9 = d = 2q$. Solving this system gives $p = 5, q = 2$, and $d = 4$.

The 2010th term is

$$p + 2009d = 5 + 2009 \cdot 4 = 8041.$$

Thus, **A** is the correct answer.

11. The solution of the equation $7^{x+7} = 8^x$ can be expressed in the form $x = \log_b 7^7$. What is b?

A $\frac{7}{15}$

B $\frac{7}{8}$

C $\frac{8}{7}$

D $\frac{15}{8}$

E $\frac{15}{7}$

Solution:

Since $x = \log_b 7^7$, we have $b^x = 7^7$.

Then

$$(7b)^x = 7^x \cdot b^x = 7^x \cdot 7^7 = 7^{x+7} = 8^x.$$

Because $x > 0$, it follows that $7b = 8$, so $b = \frac{8}{7}$.

Thus, **C** is the correct answer.

12. In a magical swamp there are two species of talking amphibians: toads, whose statements are always true, and frogs, whose statements are always false. Four amphibians, Brian, Chris, LeRoy, and Mike live together in this swamp, and they make the following statements.

Brian: "Mike and I are different species."

Chris: "LeRoy is a frog."

LeRoy: "Chris is a frog."

Mike: "Of the four of us, at least two are toads."

How many of these four amphibians are frogs?

A 0

B 1

C 2

D 3

E 4

Solution:

If Brian is a frog, then he must be lying, which means that Mike must be a frog.

If Brian is a toad, then he must be telling the truth, which also means that Mike is a frog.

Therefore, Mike is a frog, which means that Mike is lying. This means that there is at most one toad.

Then, at least one of LeRoy and Chris is a frog. This means the other is telling the truth, which makes them a toad.

This means there is one toad, which makes there be 3 frogs.

Thus, **D** is the correct answer.

13. For how many integer values of k do the graphs of $x^2 + y^2 = k^2$ and $xy = k$ not intersect?

A 0

B 1

C 2

D 4

E 8

Solution:

For $k = 0,$ the graph of $x^2 + y^2 = 0$ is the single point $(0, 0)$ and $xy = 0$ is the two axes, which meet at the origin, so the graphs intersect.

For $k \neq 0,$ the circle has radius $|k|$, and the hyperbola $xy = k$ has its two vertices nearest the origin at distance $\sqrt{2|k|}$. The graphs meet exactly when $|k| \geq \sqrt{2|k|}$, that is $|k| \geq 2.$

So they fail to intersect only when $|k| = 1,$ namely $k = 1$ and $k = -1,$ giving 2 values.

Thus, **C** is the correct answer.

14. Nondegenerate $\triangle ABC$ has integer side lengths, $\overline{BD}$ is an angle bisector, $AD = 3$, and $DC = 8$. What is the smallest possible value of the perimeter?

A 30

B 33

C 35

D 36

E 37

Solution:

Using the Angle Bisector Theorem, we have that

$$\frac{AB}{3} = \frac{BC}{8}$$

$$AB = \frac{3}{8}BC.$$

For AB and BC to be integers, we must have that BC is a multiple of 8.

To minimize the perimeter, we can set $BC = 8$ and $AB = 3$. This, however, makes the triangle degenerate.

BC must then be 16 and $AB = 6$. Since $AC = AD + DC = 11$, the perimeter is

$$16 + 6 + 11 = 33.$$

Thus, **B** is the correct answer.

15. A coin is altered so that the probability that it lands on heads is less than $\frac{1}{2}$, and when the coin is flipped four times, the probability of an equal number of heads and tails is $\frac{1}{6}$. What is the probability that the coin lands on heads?

A $\frac{\sqrt{15}-3}{6}$

B $\frac{6-\sqrt{6\sqrt{6}+2}}{12}$

C $\frac{\sqrt{2}-1}{2}$

D $\frac{3-\sqrt{3}}{6}$

E $\frac{\sqrt{3}-1}{2}$

Solution:

Let p be the probability of heads. The chance of two heads and two tails in four flips is

$$\binom{4}{2}p^2(1-p)^2 = 6p^2(1-p)^2 = \frac{1}{6}.$$

Thus $p^2(1-p)^2 = \frac{1}{36}$, so $p(1-p) = \frac{1}{6}$.

This gives $6p^2 - 6p + 1 = 0$, so $p = \frac{3 \pm \sqrt{3}}{6}$. Since $p < \frac{1}{2}$, we take $p = \frac{3-\sqrt{3}}{6}$.

Thus, **D** is the correct answer.

16. Bernardo randomly picks 3 distinct numbers from the set

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set

$$\{1, 2, 3, 4, 5, 6, 7, 8\}$$

and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

A $\frac{47}{72}$

B $\frac{37}{56}$

C $\frac{2}{3}$

D $\frac{49}{72}$

E $\frac{39}{56}$

Solution:

There are two cases: Bernardo picks a 9 or he doesn't.

Case 1: Bernardo picks a 9

Since a number is fixed, there are $\binom{8}{2} = 28$ ways to choose the other two numbers.

There are a total of $\binom{9}{3} = 84$ ways to pick all three numbers. The probability is then

$$\frac{28}{84} = \frac{1}{3}.$$

Note that if Bernardo picks a 9, he automatically has a greater number than Silvia.

This means that Bernardo always wins in this case.

Case 2: Bernardo doesn't pick a 9

There is a $1 - \frac{1}{3} = \frac{2}{3}$ chance of this happening. Since both people are choosing from the same numbers, they have an equal chance of winning.

We still need to find the probability that the numbers are the same. There is a

$$\frac{1}{\binom{8}{3}} = \frac{1}{56}$$

chance that Silvia chooses the same numbers as Bernardo. The probability that Bernardo gets a higher number is then

$$\frac{1 - \frac{1}{56}}{2} = \frac{55}{112}.$$

The total probability of Bernardo getting a higher number is then

$$\frac{1}{3} + \frac{2}{3} \cdot \frac{55}{112} = \frac{37}{56}.$$

Thus, **B** is the correct answer.

17. Equiangular hexagon $ABCDEF$ has side lengths

$$AB = CD = EF = 1$$

and

$$BC = DE = FA = r.$$

The area of $\triangle ACE$ is 70% of the area of the hexagon. What is the sum of all possible values of r?

A $\frac{4\sqrt{3}}{3}$

B $\frac{10}{3}$

C 4

D $\frac{17}{4}$

E 6

Solution:

Note that $\triangle ACE$ is equilateral. Using the Law of Cosines in $\triangle ABC$, we get $AC^2 = r^2 + 1^2 - 2r\cos 120^\circ = r^2 + r + 1.$

The area of $\triangle ACE$ is then

$$\frac{\sqrt{3}}{4}(r^2 + r + 1).$$

The three corner triangles $\triangle ABC, \triangle CDE,$ and $\triangle EFA$ each have area $\frac{1}{2} \cdot 1 \cdot r \cdot \sin 120^\circ = \frac{r\sqrt{3}}{4}.$

Thus the hexagon has area $\frac{\sqrt{3}}{4}(r^2 + r + 1) + 3 \cdot \frac{r\sqrt{3}}{4} = \frac{\sqrt{3}}{4}(r^2 + 4r + 1).$

The condition $[ACE] = 70\% \cdot [ABCDEF]$ gives

$$r^2 + r + 1 = \frac{7}{10}(r^2 + 4r + 1),$$

so $r^2 - 6r + 1 = 0$.

By Vieta's formulas, the sum of the possible values of r is 6.

Thus, **E** is the correct answer.

18. A 16-step path is to go from $(-4,-4)$ to $(4,4)$ with each step increasing either the x-coordinate or the y-coordinate by 1. How many such paths stay outside or on the boundary of the square $-2 \le x \le 2, -2 \le y \le 2$ at each step?

- A 92
- B 144
- C 1568
- D 1698
- E $12{,}800$

Solution:

Every step increases $x+y$ by 1, which runs from -8 to 8, so each path passes through exactly one lattice point with $x+y=0$.

To stay out of the open square, that point $(t,-t)$ must have $|t| \ge 2$, so it is one of $(\pm 2, \mp 2), (\pm 3, \mp 3), (\pm 4, \mp 4)$.

By symmetry consider the three points $(-4,4), (-3,3), (-2,2)$ and double. The number of paths from $(-4,-4)$ to $(-(4-j), 4-j)$ is $\binom{8}{j}$, and the number continuing on to $(4,4)$ is also $\binom{8}{j}$.

Therefore the total is

$$2\left(\binom{8}{0}^2 + \binom{8}{1}^2 + \binom{8}{2}^2\right) = 2(1 + 64 + 784) = 1698.$$

Thus, **D** is the correct answer.

19. Each of 2010 boxes in a line contains a single red marble, and for $1 \leq k \leq 2010$, the box in the kth position also contains k white marbles. Isabella begins at the first box and successively draws a single marble at random from each box, in order. She stops when she first draws a red marble. Let $P(n)$ be the probability that Isabella stops after drawing exactly n marbles. What is the smallest value of n for which $P(n) < \frac{1}{2010}$?

A 45

B 63

C 64

D 201

E 1005

Solution:

Since there are $k+1$ marbles in the k th box, there is a $\frac{k}{k+1}$ chance Isabella draws a white marble from it.

The probability of drawing a red marble is then $\frac{1}{k+1}$. To stop after drawing the n th marble, the first $n-1$ marbles must have been white.

This happens with a probability of

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \ldots \cdot \frac{n-1}{n} \cdot \frac{1}{n+1}.$$

Note that all the numerators cancel with the adjacent denominator, which means that this expression reduces to $\frac{1}{n(n+1)}$.

We have to find the smallest n such that

$$\frac{1}{n(n+1)} < \frac{1}{2010}$$

$$n(n+1) > 2010.$$

Guessing and checking gives us that the smallest n that works is 45.

Thus, **A** is the correct answer.

20. Arithmetic sequences (a_n) and (b_n) have integer terms with $a_1 = b_1 = 1 < a_2 \le b_2$ and $a_n b_n = 2010$ for some n. What is the largest possible value of n?

A 2

B 3

C 8

D 288

E 2009

Solution:

Since $a_n = 1 + (n-1)d_1$ and $b_n = 1 + (n-1)d_2$ for integers d_1, d_2, the value $n - 1$ divides both $a_n - 1$ and $b_n - 1$, hence divides $\gcd(a_n - 1, b_n - 1)$.

The factor pairs of 2010 with $2 \le a_n \le b_n$ are $(2, 1005), (3, 670), (5, 402), (6, 335), (10, 201), (15, 134)$, and $(30, 67)$.

For every pair except $(15, 134)$, the numbers $a_n - 1$ and $b_n - 1$ are relatively prime, forcing $n = 2$. For $(15, 134)$, $\gcd(14, 133) = 7$, so $n - 1$ can equal 7, giving $n = 8$.

The sequences $a_n = 2n - 1$ and $b_n = 19n - 18$ realize this, so the largest value is 8.

Thus, **C** is the correct answer.

21. The graph of $y = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2$ lies above the line $y = bx + c$ except at three values of x, where the graph and the line intersect. What is the largest of those values?

- A 4
- B 5
- C 6
- D 7
- E 8

Solution:

Let $f(x)$ be the graph minus the line. It is nonnegative and vanishes at three points, each a double root, so

$$f(x) = \left(x^3 - Ax^2 + Bx - C\right)^2.$$

Matching coefficients gives $-2A = -10 \Rightarrow A = 5$, then $A^2 + 2B = 29 \Rightarrow B = 2$, then $-2C - 2AB = -4 \Rightarrow C = -8$.

Thus the cubic is $x^3 - 5x^2 + 2x + 8 = (x+1)(x-2)(x-4)$, with roots $-1, 2,$ and 4. The largest is 4.

Thus, **A** is the correct answer.

22. What is the minimum value of

$$f(x) = |x-1| + |2x-1| + |3x-1| + \cdots + |119x-1|?$$

A 49

B 50

C 51

D 52

E 53

Solution:

The function f is piecewise linear with breakpoints at $x = \frac{1}{k}$. On the interval $\left[\frac{1}{m}, \frac{1}{m-1}\right]$ its slope is

$$\sum_{k=m}^{119} k - \sum_{k=1}^{m-1} k = 7140 - (m-1)m,$$

where $7140 = \frac{119 \cdot 120}{2}$.

This slope is zero when $(m-1)m = 7140$, i.e. $m = 85$, so the minimum occurs at the right endpoint $x = \frac{1}{84}$.

There, terms with $k \leq 84$ contribute $\frac{84-k}{84}$ and terms with $k \geq 85$ contribute $\frac{k-84}{84}$, so

$$f\left(\tfrac{1}{84}\right) = \frac{3486}{84} + \frac{630}{84} = 41.5 + 7.5 = 49.$$

Thus, **A** is the correct answer.

23. The number obtained from the last two nonzero digits of $90!$ is equal to n. What is n?

A 12

B 32

C 48

D 52

E 68

Solution:

The number of trailing zeroes in $90!$ is $\left\lfloor \frac{90}{5} \right\rfloor + \left\lfloor \frac{90}{25} \right\rfloor = 21$. Let $N = \frac{90!}{10^{21}}$.

There are still more than two factors of 2 left after removing 10^{21}, so $N \equiv 0 \pmod{4}$.

Let A be the product of factors of $90!$ not divisible by 5, and let B be the product of the factors divisible by 5. Grouping residues modulo 25 gives $A \equiv 1 \pmod{25}$ and $\frac{B}{5^{21}} \equiv -1 \pmod{25}$.

Therefore $\frac{90!}{5^{21}} \equiv -1 \pmod{25}$. Since $2^{21} \equiv 2 \pmod{25}$, $N = \frac{90!}{5^{21} \cdot 2^{21}} \equiv -13 \equiv 12 \pmod{25}$.

The number congruent to $0 \pmod{4}$ and $12 \pmod{25}$ is $12 \pmod{100}$, so the last two nonzero digits form 12.

Thus, **A** is the correct answer.

24. Let $f(x) = \log_{10}\left(\sin(\pi x) \cdot \sin(2\pi x) \cdot \sin(3\pi x) \cdots \sin(8\pi x)\right)$. The intersection of the domain of $f(x)$ with the interval $[0, 1]$ is a union of n disjoint open intervals. What is n?

- A 2
- B 12
- C 18
- D 22
- E 36

Solution:

Let $g(x) = \prod_{k=1}^{8} \sin(k\pi x)$; the domain of f is where $g(x) > 0$. Since $\sin(k\pi(1 - x)) = (-1)^{k+1}\sin(k\pi x)$ and $\sum_{k=1}^{8}(k+1)$ is even, $g(1-x) = g(x)$, so it suffices to study $\left(0, \frac{1}{2}\right)$ and double.

In $\left(0, \frac{1}{2}\right)$ the zeros of g are the fractions $\frac{k}{n}$ with $2 \le n \le 8, 1 \le k < \frac{n}{2}$, and $\gcd(k, n) = 1$. For $n = 2, \ldots, 8$ there are $0, 1, 1, 2, 1, 3, 2$ of them, totaling 10.

These 10 zeros split $\left(0, \frac{1}{2}\right)$ into 11 subintervals on which g has constant sign. Near 0 every factor is positive, so $g > 0$ there, and the sign flips at each zero except $\frac{1}{4} = \frac{2}{8}$ and $\frac{1}{3} = \frac{2}{6}$, where an even number of factors vanish.

Tracking the signs, exactly 6 of the 11 subintervals have $g > 0$. By symmetry there are 6 more in $\left(\frac{1}{2}, 1\right)$, so $n = 12$.

Thus, **B** is the correct answer.

25. Two quadrilaterals are considered the same if one can be obtained from the other by a rotation and a translation. How many different convex cyclic quadrilaterals are there with integer sides and perimeter equal to 32?

A 560

B 564

C 568

D 1498

E 2255

Solution:

A convex cyclic quadrilateral is determined up to rotation and translation by its cyclic sequence of side lengths, and it exists exactly when the largest side is less than the sum of the others. With perimeter 32, this means each side is at most 15.

First count ordered quadruples (a, b, c, d) of positive integers with $a + b + c + d = 32$ and each entry at most 15. Without the upper bound there are $\binom{31}{3} = 4495$; removing those with some entry at least 16 subtracts $4\binom{16}{3} = 2240$, leaving 2255.

Rotations of the quadrilateral correspond to cyclic permutations of (a, b, c, d). By Burnside's lemma the number of distinct quadrilaterals is

$$\frac{1}{4}\left(2255 + f_1 + f_2 + f_3\right),$$

where f_i counts quadruples fixed by rotating i positions.

A one- or three-step rotation fixes only $(8, 8, 8, 8)$, so $f_1 = f_3 = 1$. A two-step rotation fixes (a, b, a, b) with $a + b = 16$ and $1 \leq a, b \leq 15$, giving $f_2 = 15$.

Hence the count is

$$\frac{1}{4}(2255 + 1 + 15 + 1) = \frac{2272}{4} = 568.$$

Thus, **C** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc12/2010A