2009 AMC 12B Solutions

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1. Each morning of her five-day workweek, Jane bought either a 50-cent muffin or a 75-cent bagel. Her total cost for the week was a whole number of dollars. How many bagels did she buy?

A 1

B 2

C 3

D 4

E 5

Solution:

With b bagels she buys $5-b$ muffins, costing $50(5-b)+75b=250+25b$ cents. For a whole number of dollars this must be a multiple of 100, so $25b$ must end in 50, meaning b is even.

Among $b=2$ and $b=4$, only $b=2$ works: $250+50=300$ cents $=\$3.$

Thus, the correct answer is **B**.

2. Paula the painter had just enough paint for 30 identically sized rooms. Unfortunately, on the way to work, three cans of paint fell off her truck, so she had only enough paint for 25 rooms. How many cans of paint did she use for the 25 rooms?

- A 10
- B 12
- C 15
- D 18
- E 25

Solution:

Losing 3 cans cost her 5 rooms, so 3 cans paint 5 rooms and each room needs $\frac{3}{5}$ of a can.

For 25 rooms she used $25 \cdot \frac{3}{5} = 15$ cans.

Thus, the correct answer is **C**.

3. Twenty percent less than 60 is one-third more than what number?

- A 16
- B 30
- C 32
- D 36
- E 48

Solution:

Twenty percent less than 60 is $0.8 \cdot 60 = 48$.

One-third more than n is $\frac{4}{3}n$, so $\frac{4}{3}n = 48$ gives $n = 36$.

Thus, the correct answer is **D**.

4. A rectangular yard contains two flower beds in the shape of congruent isosceles right triangles. The remainder of the yard has a trapezoidal shape, as shown. The parallel sides of the trapezoid have lengths 15 and 25 meters. What fraction of the yard is occupied by the flower beds?

A $\frac{1}{8}$

B $\frac{1}{6}$

C $\frac{1}{5}$

D $\frac{1}{4}$

E $\frac{1}{3}$

Solution:

The parallel sides differ by $25 - 15 = 10$, so each triangle has legs $\frac{10}{2} = 5$ and area $\frac{1}{2} \cdot 5^2 = \frac{25}{2}$. The two beds total 25.

The rectangle measures 25 by 5, so its area is 125, and the fraction occupied is $\frac{25}{125} = \frac{1}{5}$.

Thus, the correct answer is **C**.

5. Kiana has two older twin brothers. The product of their three ages is 128. What is the sum of their three ages?

- A 10
- B 12
- C 16
- D 18
- E 24

Solution:

Since $128 = 2^7$, each age is a power of 2. The twins share an age t, so Kiana's age is $\frac{128}{t^2}$.

Taking $t = 8$ gives Kiana $\frac{128}{64} = 2$, who is younger than the twins. (Smaller twins would make Kiana older, which is not allowed.) The sum is $8 + 8 + 2 = 18$.

Thus, the correct answer is **D**.

6. By inserting parentheses, it is possible to give the expression

$$2 \times 3 + 4 \times 5$$

several values. How many different values can be obtained?

- A 2
- B 3
- C 4
- D 5
- E 6

Solution:

The genuinely different groupings give

$(2 \times 3) + (4 \times 5) = 26,\ ((2 \times 3) + 4) \times 5 = 50,\ 2 \times (3 + (4 \times 5)) = 46$, and $2 \times (3 + 4) \times 5 = 70.$

These are all distinct, so 4 values can be obtained.

Thus, the correct answer is **C**.

7. In a certain year the price of gasoline rose by 20% during January, fell by 20% during February, rose by 25% during March, and fell by $x\%$ during April. The price of gasoline at the end of April was the same as it had been at the beginning of January. To the nearest integer, what is x?

- A 12
- B 17
- C 20
- D 25
- E 35

Solution:

After January through March the price is $1.2 \cdot 0.8 \cdot 1.25 = 1.2$ times the original.

To return to the original, April must multiply by $\frac{1}{1.2}$, a decrease of $1 - \frac{1}{1.2} = \frac{1}{6} \approx 16.7\%$. To the nearest integer, $x = 17$.

Thus, the correct answer is **B**.

8. When a bucket is two-thirds full of water, the bucket and water weigh a kilograms. When the bucket is one-half full of water the total weight is b kilograms. In terms of a and b, what is the total weight in kilograms when the bucket is full of water?

A $\frac{2}{3}a + \frac{1}{3}b$

B $\frac{3}{2}a - \frac{1}{2}b$

C $\frac{3}{2}a + b$

D $\frac{3}{2}a + 2b$

E $3a - 2b$

Solution:

Let x be the bucket's weight and y the weight of a full bucket of water. Then $x + \frac{2}{3}y = a$ and $x + \frac{1}{2}y = b$.

Subtracting gives $\frac{1}{6}y = a - b$, so $y = 6a - 6b$ and $x = b - \frac{1}{2}y = 4b - 3a$. The full weight is $x + y = 3a - 2b$.

Thus, the correct answer is **E**.

9. Triangle ABC has vertices $A = (3, 0), B = (0, 3)$, and C, where C is on the line $x + y = 7$. What is the area of $\triangle ABC$?

- A 6
- B 8
- C 10
- D 12
- E 14

Solution:

Line AB has equation $x + y = 3$, which is parallel to $x + y = 7$, so the area is independent of where C lies on that line.

Take $C = (7, 0)$. Then the base $AC = 4$ lies on the x-axis with height 3, giving area $\frac{1}{2} \cdot 4 \cdot 3 = 6$.

Thus, the correct answer is **A**.

10. A particular 12-hour digital clock displays the hour and minute of a day. Unfortunately, whenever it is supposed to display a 1, it mistakenly displays a 9. For example, when it is 1:16 pm the clock incorrectly shows 9:96 pm. What fraction of the day will the clock show the correct time?

A $\frac{1}{2}$

B $\frac{5}{8}$

C $\frac{3}{4}$

D $\frac{5}{6}$

E $\frac{9}{10}$

Solution:

The hours containing a 1 are $1, 10, 11, 12$, so 8 of the 12 hours display correctly, a fraction $\frac{2}{3}$.

A minute is wrong if either digit is 1: the tens digit gives 10-19 (10 minutes), and the ones digit adds $01, 21, 31, 41, 51$ (5 more), 15 in all. So $\frac{45}{60} = \frac{3}{4}$ of minutes are correct.

The fraction of the day is $\frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$.

Thus, the correct answer is **A**.

11. On Monday, Millie puts a quart of seeds, 25% of which are millet, into a bird feeder. On each successive day she adds another quart of the same mix of seeds without removing any seeds that are left. Each day the birds eat only 25% of the millet in the feeder, but they eat all of the other seeds. On which day, just after Millie has placed the seeds, will the birds find that more than half the seeds in the feeder are millet?

- A Tuesday
- B Wednesday
- C Thursday
- D Friday
- E Saturday

Solution:

Each day the birds leave $\frac{3}{4}$ of the millet and Millie adds $\frac{1}{4}$ quart of new millet, so after n days the millet is

$$\frac{1}{4}\left(1+\frac{3}{4}+\cdots+\left(\frac{3}{4}\right)^{n-1}\right)=1-\left(\frac{3}{4}\right)^{n}$$

quart.

The non-millet seeds always total $\frac{3}{4}$ quart, so millet exceeds half when $1-\left(\frac{3}{4}\right)^{n}>\frac{3}{4}$, that is, $\left(\frac{3}{4}\right)^{n}<\frac{1}{4}$.

Since $\left(\frac{3}{4}\right)^{4}=\frac{81}{256}>\frac{1}{4}$ and $\left(\frac{3}{4}\right)^{5}=\frac{243}{1024}<\frac{1}{4}$, this first happens on day 5, which is Friday.

Thus, the correct answer is **D**.

12. The fifth and eighth terms of a geometric sequence of real numbers are $7!$ and $8!$ respectively. What is the first term?

- A 60
- B 75
- C 120
- D 225
- E 315

Solution:

The eighth term divided by the fifth term is $r^3 = \frac{8!}{7!} = 8$, so $r = 2$.

The fifth term is $ar^4 = 7!$, so $a = \frac{7!}{16} = \frac{5040}{16} = 315$.

Thus, the correct answer is **E**.

13. Triangle ABC has $AB = 13$ and $AC = 15$, and the altitude to BC has length 12. What is the sum of the two possible values of BC?

- A 15
- B 16
- C 17
- D 18
- E 19

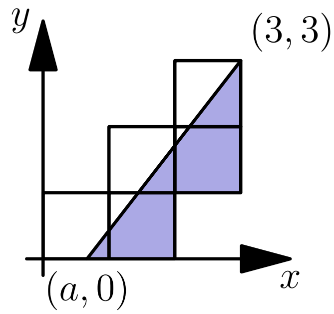
Solution:

Let D be the foot of the altitude from A. Then $BD = \sqrt{13^2 - 12^2} = 5$ and $DC = \sqrt{15^2 - 12^2} = 9$.

If D lies between B and C, then $BC = 5 + 9 = 14$; if the triangle is obtuse, $BC = 9 - 5 = 4$. The sum is $14 + 4 = 18$.

Thus, the correct answer is **D**.

14. Five unit squares are arranged in the coordinate plane as shown, with the lower left corner at the origin. The slanted line, extending from $(a, 0)$ to $(3, 3)$, divides the entire region into two regions of equal area. What is a?



A $\frac{1}{2}$

B $\frac{3}{5}$

C $\frac{2}{3}$

D $\frac{3}{4}$

E $\frac{4}{5}$

Solution:

The five squares have total area 5, so each region must have area $\frac{5}{2}$.

The line from $(a, 0)$ to $(3, 3)$ together with the axes bounds a triangle of base $3 - a$ and height 3; the region on the lower-right side of the line is this triangle with one unit square removed. Setting

$$\frac{3(3-a)}{2} - 1 = \frac{5}{2}$$

gives $3(3 - a) = 7$, so $a = \frac{2}{3}$.

Thus, the correct answer is **C**.

15. Assume $0 < r < 3$. Below are five equations for x. Which equation has the largest solution x?

A $3(1+r)^x = 7$

B $3(1+r/10)^x = 7$

C $3(1+2r)^x = 7$

D $3(1+\sqrt{r})^x = 7$

E $3(1+1/r)^x = 7$

Solution:

Each equation gives $x = \dfrac{\log(7/3)}{\log(1+f(r))}$, which is largest when the positive quantity $f(r)$ is smallest.

For $0 < r < 3$, among $r,\ \dfrac{r}{10},\ 2r,\ \sqrt{r},\ \dfrac{1}{r}$, the smallest is $\dfrac{r}{10}$: it is below r and below $\sqrt{r}$ since $r < 3 < 100$. So equation (B) has the largest solution.

Thus, the correct answer is **B**.

16. Trapezoid $ABCD$ has $AD \parallel BC, BD = 1, \angle DBA = 23^\circ$, and $\angle BDC = 46^\circ$. The ratio $BC : AD$ is $9 : 5$. What is CD?

A $\frac{7}{9}$

B $\frac{4}{5}$

C $\frac{13}{15}$

D $\frac{8}{9}$

E $\frac{14}{15}$

Solution:

Draw the line through D parallel to AB, meeting BC at E, so $ABED$ is a parallelogram with $BE = AD$. Then $\angle BDE = \angle DBA = 23^\circ$, and since $\angle BDC = 46^\circ$, segment DE bisects $\angle BDC$.

By the angle bisector theorem in $\triangle BDC$, $\frac{EC}{BE} = \frac{DC}{DB}$, so

$$CD = DB \cdot \frac{BC - AD}{AD} = 1 \cdot \left(\frac{9}{5} - 1\right) = \frac{4}{5}.$$

Thus, the correct answer is **B**.

17. Each face of a cube is given a single narrow stripe painted from the center of one edge to the center of its opposite edge. The choice of the edge pairing is made at random and independently for each face. What is the probability that there is a continuous stripe encircling the cube?

A $\frac{1}{8}$

B $\frac{3}{16}$

C $\frac{1}{4}$

D $\frac{3}{8}$

E $\frac{1}{2}$

Solution:

Each of the 6 faces has 2 equally likely stripe orientations, for $2^6 = 64$ configurations.

An encircling stripe runs around one of the 3 pairs of opposite faces. Fixing such a band, the four faces it passes through must be aligned, with probability $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$, while the two remaining faces are free. The 3 possible bands are disjoint events, so the probability is $3 \cdot \frac{1}{16} = \frac{3}{16}$.

Thus, the correct answer is **B**.

18. Rachel and Robert run on a circular track. Rachel runs counterclockwise and completes a lap every 90 seconds, and Robert runs clockwise and completes a lap every 80 seconds. Both start from the start line at the same time. At some random time between 10 minutes and 11 minutes after they begin to run, a photographer standing inside the track takes a picture that shows one-fourth of the track, centered on the starting line. What is the probability that both Rachel and Robert are in the picture?

A $\frac{1}{16}$

B $\frac{1}{8}$

C $\frac{3}{16}$

D $\frac{1}{4}$

E $\frac{5}{16}$

Solution:

The picture covers the arc within $\frac{1}{8}$ lap of the start on each side. At 600 s Rachel has run $6\frac{2}{3}$ laps, 30 s short of the line; a quarter lap takes her 22.5 s, so she is in view between $30 - 11.25 = 18.75$ s and $30 + 11.25 = 41.25$ s of the 10th minute.

At 600 s Robert is 40 s from the line; a quarter lap takes 20 s, so he is in view between 30 and 50 s. Both appear between 30 and 41.25 s, a window of 11.25 s out of 60, giving probability $\frac{11.25}{60} = \frac{3}{16}$.

Thus, the correct answer is **C**.

19. For each positive integer n, let $f(n) = n^4 - 360n^2 + 400$. What is the sum of all values of $f(n)$ that are prime numbers?

A 794

B 796

C 798

D 800

E 802

Solution:

Write

$$f(n) = n^4 + 40n^2 + 400 - 400n^2 = (n^2 + 20)^2 - (20n)^2 = (n^2 + 20n + 20)(n^2 - 20n + 20).$$

For $f(n)$ to be prime the smaller factor must be 1: solving $n^2 - 20n + 20 = 1$ gives $(n - 1)(n - 19) = 0$, so $n = 1$ or $n = 19$.

Then $f(1) = 41$ and $f(19) = 761$ are both prime, summing to 802.

Thus, the correct answer is **E**.

20. A convex polyhedron Q has vertices $V_1, V_2, \ldots, V_n$, and 100 edges. The polyhedron is cut by planes $P_1, P_2, \ldots, P_n$ in such a way that plane P_k cuts only those edges that meet at vertex V_k. In addition, no two planes intersect inside or on Q. The cuts produce n pyramids and a new polyhedron R. How many edges does R have?

A 200

B $2n$

C 300

D 400

E $4n$

Solution:

Each of the 100 edges is cut once near each endpoint, so R has $2 \cdot 100 = 200$ vertices.

The cut at vertex V_k creates a small polygon whose number of edges equals the degree of V_k; summed over all vertices this is 200, the total number of edge-endpoints. The middle portion of each original edge also survives, adding 100 edges. So R has $200 + 100 = 300$ edges.

Thus, the correct answer is **C**.

21. Ten women sit in 10 seats in a line. All of the 10 get up and then reseat themselves using all 10 seats, each sitting in the seat she was in before or a seat next to the one she occupied before. In how many ways can the women be reseated?

A 89

B 90

C 120

D 210

E 2^{38}

Solution:

Let S_n be the number of valid reseatings of n women. The rightmost woman either keeps her seat, leaving S_{n-1} ways for the rest, or swaps with her left neighbor — the only other way to fill the end seat — leaving S_{n-2} ways.

Thus $S_n = S_{n-1} + S_{n-2}$ with $S_1 = 1$ and $S_2 = 2$, giving the Fibonacci values $1, 2, 3, 5, 8, 13, 21, 34, 55, 89$. So $S_{10} = 89$.

Thus, the correct answer is **A**.

22. Parallelogram $ABCD$ has area $1{,}000{,}000$. Vertex A is at $(0, 0)$ and all other vertices are in the first quadrant. Vertices B and D are lattice points on the lines $y = x$ and $y = kx$ for some integer $k > 1$, respectively. How many such parallelograms are there?

- A 49
- B 720
- C 784
- D 2009
- E 2048

Solution:

Let $B = (b, b)$ and $D = (d, kd)$ with b, d, k positive integers and $k > 1$. The area is $(k-1)bd = 1{,}000{,}000 = 2^6 \cdot 5^6$.

Each parallelogram corresponds to an ordered triple $(k-1, b, d)$ of positive integers with product $2^6 \cdot 5^6$. The six 2's distribute among the three factors in $\binom{6+2}{2} = 28$ ways, and likewise the six 5's in 28 ways, giving $28^2 = 784$.

Thus, the correct answer is **C**.

23. A region S in the complex plane is defined by

$$S = \{x + iy : -1 \le x \le 1,\ -1 \le y \le 1\}.$$

A complex number $z = x + iy$ is chosen uniformly at random from S. What is the probability that $\left(\frac{3}{4} + \frac{3}{4}i\right) z$ is also in S?

A $\frac{1}{2}$

B $\frac{2}{3}$

C $\frac{3}{4}$

D $\frac{7}{9}$

E $\frac{7}{8}$

Solution:

Expanding, $\left(\frac{3}{4} + \frac{3}{4}i\right)(x + iy) = \frac{3}{4}(x - y) + \frac{3}{4}(x + y)i$. Both parts lie in $[-1, 1]$ iff $|x - y| \le \frac{4}{3}$ and $|x + y| \le \frac{4}{3}$.

Within the square S (area 4) these fail only in four corner triangles. Near $(1, 1)$, the line $x + y = \frac{4}{3}$ cuts off a right triangle with legs $\frac{2}{3}$, area $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{2}{9}$.

The four corners remove $4 \cdot \frac{2}{9} = \frac{8}{9}$, leaving $4 - \frac{8}{9} = \frac{28}{9}$. The probability is $\frac{28/9}{4} = \frac{7}{9}$.

Thus, the correct answer is **D**.

24. For how many values of x in $[0, \pi]$ is $\sin^{-1}(\sin 6x) = \cos^{-1}(\cos x)$?

Note: The functions $\sin^{-1} = \arcsin$ and $\cos^{-1} = \arccos$ denote inverse trigonometric functions.

- A 3
- B 4
- C 5
- D 6
- E 7

Solution:

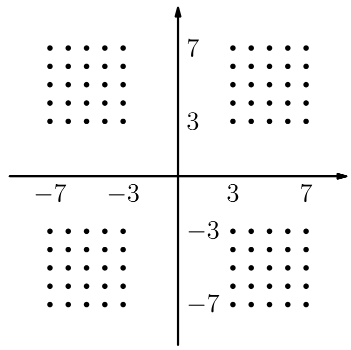
On $[0, \pi],\ \cos^{-1}(\cos x) = x$. Since $\sin^{-1}$ takes values in $[-\frac{\pi}{2}, \frac{\pi}{2}]$, any solution requires $x \in [0, \frac{\pi}{2}]$, where the equation becomes $\sin 6x = \sin x$.

As x goes from 0 to $\frac{\pi}{2}$, $\sin 6x$ runs $0 \to 1 \to -1 \to 1 \to 0$ (peaks at $\frac{\pi}{12}, \frac{5\pi}{12}$, trough at $\frac{\pi}{4}$), while $\sin x$ increases from 0 to 1.

Besides $x = 0$, the graphs cross once in each of $[\frac{\pi}{12}, \frac{\pi}{4}]$, $[\frac{\pi}{4}, \frac{5\pi}{12}]$, and $[\frac{5\pi}{12}, \frac{\pi}{2}]$, for 4 solutions in all.

Thus, the correct answer is **B**.

25. The set G is defined by the points (x, y) with integer coordinates, $3 \le |x| \le 7$, and $3 \le |y| \le 7$. How many squares of side at least 6 have their four vertices in G?



A 125

B 150

C 175

D 200

E 225

Solution:

G consists of four 5×5 blocks $G_1, \ldots, G_4$, one in each quadrant. Any square of side ≥ 6 uses exactly one vertex in each block, since two points in one block are less than 6 apart while points in different blocks are at least 6 apart.

Sliding each block inward by $(\pm 5, \pm 5)$ superimposes them on one 5×5 grid G' (points with $|x|, |y| \le 2$). Each such square maps to either a single point of G' or a square in G'. So the count equals the number of points of G' plus 4 times the number of squares with vertices in G'.

A 5×5 grid has 25 points and 50 squares of all tilts, so the total is $25 + 4 \cdot 50 = 225$.

Thus, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/2009B