2008 AMC 12B Solutions

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1. A basketball player made 5 baskets during a game. Each basket was worth either 2 or 3 points. How many different numbers could represent the total points scored by the player?

   A 2

   B 3

   C 4

   D 5

   E 6

Solution:

The total ranges from $5 \cdot 2 = 10$ (all two-pointers) to $5 \cdot 3 = 15$ (all three-pointers). Swapping one two-pointer for a three-pointer raises the total by exactly 1, so every integer in between occurs.

The possible totals are $10, 11, 12, 13, 14, 15$, which is 6 values.

Thus, the correct answer is **E**.

2. A 4×4 block of calendar dates is shown. The order of the numbers in the second row is to be reversed. Then the order of the numbers in the fourth row is to be reversed. Finally, the numbers on each diagonal are to be added. What will be the positive difference between the two diagonal sums?

1	2	3	4
8	9	10	11
15	16	17	18
22	23	24	25

A 2

B 4

C 6

D 8

E 10

Solution:

Reversing the second and fourth rows gives the array with rows $1\,2\,3\,4$, $11\,10\,9\,8$, $15\,16\,17\,18$, and $25\,24\,23\,22$.

The main diagonal sums to $1 + 10 + 17 + 22 = 50$, and the other diagonal sums to $4 + 9 + 16 + 25 = 54$.

The positive difference is $54 - 50 = 4$.

Thus, the correct answer is **B**.

3. A semipro baseball league has teams with 21 players each. League rules state that a player must be paid at least $\$15{,}000$, and that the total of all players' salaries for each team cannot exceed $\$700{,}000$. What is the maximum possible salary, in dollars, for a single player?

- A $270{,}000$
- B $385{,}000$
- C $400{,}000$
- D $430{,}000$
- E $700{,}000$

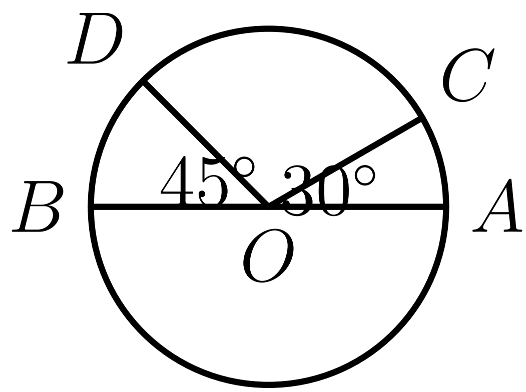
Solution:

One player earns the most when the other 20 players each receive the minimum salary of $\$15{,}000$.

Thus the maximum salary is $\$700{,}000 - 20 \cdot \$15{,}000 = \$700{,}000 - \$300{,}000 = \$400{,}000$.

Thus, the correct answer is **C**.

4. On circle O, points C and D are on the same side of diameter $\overline{AB}, \angle AOC = 30^{\circ}$, and $\angle DOB = 45^{\circ}$. What is the ratio of the area of the smaller sector COD to the area of the circle?



A $\frac{2}{9}$

B $\frac{1}{4}$

C $\frac{5}{18}$

D $\frac{7}{24}$

E $\frac{3}{10}$

Solution:

Since $\angle AOC, \angle COD$, and $\angle DOB$ fill the straight angle over diameter $\overline{AB}$,

$$\angle COD = 180^{\circ} - 30^{\circ} - 45^{\circ} = 105^{\circ}.$$

The sector's share of the circle is $\frac{105^{\circ}}{360^{\circ}} = \frac{7}{24}$.

Thus, the correct answer is **D**.

5. A class collects $\$50$ to buy flowers for a classmate who is in the hospital. Roses cost $\$3$ each, and carnations cost $\$2$ each. No other flowers are to be used. How many different bouquets could be purchased for exactly $\$50$?

A 1

B 7

C 9

D 16

E 17

Solution:

Let r be the number of roses and c the number of carnations, so $3r + 2c = 50$ with $r, c \geq 0$.

Because $2c$ and 50 are even, $3r$ must be even, forcing r to be even. The largest possible r is 16 (since $3 \cdot 17 > 50$), so $r \in \{0, 2, 4, \ldots, 16\}$.

That gives 9 values of r, each determining a bouquet.

Thus, the correct answer is **C**.

6. Postman Pete has a pedometer to count his steps. The pedometer records up to 99999 steps, then flips over to 00000 on the next step. Pete plans to determine his mileage for a year. On January 1 Pete sets the pedometer to $00000.$ During the year, the pedometer flips from 99999 to 00000 forty-four times. On December 31 the pedometer reads $50000.$ Pete takes 1800 steps per mile. Which of the following is closest to the number of miles Pete walked during the year?

A 2500

B 3000

C 3500

D 4000

E 4500

Solution:

Each flip counts 100000 steps, so the year's steps total

$$44 \cdot 100000 + 50000 = 4{,}450{,}000.$$

At 1800 steps per mile, the mileage is $\dfrac{4{,}450{,}000}{1800} \approx 2472,$ which is closest to $2500.$

Thus, the correct answer is **A**.

7. For real numbers a and b, define $a\$b = (a-b)^2$. What is $(x-y)^2\$(y-x)^2$?

A 0

B $x^2 + y^2$

C $2x^2$

D $2y^2$

E $4xy$

Solution:

Since $(y-x)^2 = (x-y)^2$, both inputs to the operation are the same value $t = (x-y)^2$.

Therefore $(x-y)^2\$(y-x)^2 = t\$t = (t-t)^2 = 0$.

Thus, the correct answer is **A**.

8. Points B and C lie on $\overline{AD}$. The length of $\overline{AB}$ is 4 times the length of $\overline{BD}$, and the length of $\overline{AC}$ is 9 times the length of $\overline{CD}$. The length of $\overline{BC}$ is what fraction of the length of $\overline{AD}$?

A $\frac{1}{36}$

B $\frac{1}{13}$

C $\frac{1}{10}$

D $\frac{5}{36}$

E $\frac{1}{5}$

Solution:

Since $AB = 4\,BD$ and $AB + BD = AD$, we have $5\,BD = AD$, so $BD = \frac{1}{5}AD$.

Likewise $AC = 9\,CD$ with $AC + CD = AD$ gives $CD = \frac{1}{10}AD$.

Because B and C both measure from A, $BC = BD - CD = \frac{1}{5}AD - \frac{1}{10}AD = \frac{1}{10}AD$.

Thus, the correct answer is **C**.

9. Points A and B are on a circle of radius 5 and $AB = 6$. Point C is the midpoint of the minor arc AB. What is the length of the line segment AC?

- A $\sqrt{10}$
- B $\frac{7}{2}$
- C $\sqrt{14}$
- D $\sqrt{15}$
- E 4

Solution:

Let O be the center and D the point where $\overline{OC}$ meets $\overline{AB}$. Since C is the midpoint of arc $AB, \overline{OC}$ is the perpendicular bisector of the chord, so $AD = 3$.

In right triangle $ADO, OD = \sqrt{5^2 - 3^2} = 4$, so $DC = OC - OD = 5 - 4 = 1$.

Then in right triangle $ADC, AC = \sqrt{AD^2 + DC^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$.

Thus, the correct answer is **A**.

10. Bricklayer Brenda would take 9 hours to build a chimney alone, and bricklayer Brandon would take 10 hours to build it alone. When they work together, they talk a lot, and their combined output is decreased by 10 bricks per hour. Working together, they build the chimney in 5 hours. How many bricks are in the chimney?

A 500

B 900

C 950

D 1000

E 1900

Solution:

Let n be the number of bricks. Alone, Brenda lays $\frac{n}{9}$ bricks per hour and Brandon lays $\frac{n}{10}$. Together, their rate is $\frac{n}{9} + \frac{n}{10} - 10.$

Working for 5 hours completes the chimney:

$$5\left(\frac{n}{9} + \frac{n}{10} - 10\right) = n.$$

Expanding, $\frac{5n}{9} + \frac{5n}{10} - 50 = n$, so $\frac{95n}{90} - n = 50$, giving $\frac{5n}{90} = 50.$

Hence $n = 900.$

Thus, the correct answer is **B**.

11. A cone-shaped mountain has its base on the ocean floor and has a height of 8000 feet. The top $\frac{1}{8}$ of the volume of the mountain is above water. What is the depth of the ocean at the base of the mountain, in feet?

A 4000

B $2000(4-\sqrt{2})$

C 6000

D 6400

E 7000

Solution:

The part above the water is a cone similar to the whole mountain, with volume $\frac{1}{8}$ of the total. Since volume scales as the cube of length, the above-water cone's height is $\sqrt[3]{\frac{1}{8}} = \frac{1}{2}$ of the full height.

So the above-water height is $8000 \cdot \frac{1}{2} = 4000$ feet.

The ocean depth at the base is the submerged height, $8000 - 4000 = 4000$ feet.

Thus, the correct answer is **A**.

12. For each positive integer n, the mean of the first n terms of a sequence is n. What is the 2008th term of the sequence?

- A 2008
- B 4015
- C 4016
- D $4{,}030{,}056$
- E $4{,}032{,}064$

Solution:

Since the mean of the first n terms is n, their sum is $n \cdot n = n^2$.

The nth term is the difference of consecutive sums, $n^2 - (n-1)^2 = 2n - 1$.

For $n = 2008$, the term is $2 \cdot 2008 - 1 = 4015$.

Thus, the correct answer is **B**.

13. Vertex E of equilateral $\triangle ABE$ is in the interior of unit square $ABCD$. Let R be the region consisting of all points inside $ABCD$ and outside $\triangle ABE$ whose distance from $\overline{AD}$ is between $\frac{1}{3}$ and $\frac{2}{3}$. What is the area of R?

A $\dfrac{12-5\sqrt{3}}{72}$

B $\dfrac{12-5\sqrt{3}}{36}$

C $\dfrac{\sqrt{3}}{18}$

D $\dfrac{3-\sqrt{3}}{9}$

E $\dfrac{\sqrt{3}}{12}$

Solution:

Place $A=(0,0), B=(1,0), C=(1,1), D=(0,1)$, so $\overline{AD}$ lies along the y-axis and distance from $\overline{AD}$ is the x-coordinate. The region lies in the strip $\frac{1}{3} \le x \le \frac{2}{3}$, which within the square has area $\frac{1}{3}$.

Equilateral $\triangle ABE$ has $E=\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, with side AE on $y=\sqrt{3}\,x$ and side BE on $y=\sqrt{3}(1-x)$. The area of the triangle inside the strip is

$$\int_{1/3}^{1/2} \sqrt{3}\,x\,dx + \int_{1/2}^{2/3} \sqrt{3}(1-x)\,dx = 2\int_{1/3}^{1/2} \sqrt{3}\,x\,dx = \frac{5\sqrt{3}}{36}.$$

Therefore

$$[R] = \frac{1}{3} - \frac{5\sqrt{3}}{36} = \frac{12-5\sqrt{3}}{36}.$$

Thus, the correct answer is **B**.

14. A circle has a radius of $\log_{10}(a^2)$ and a circumference of $\log_{10}(b^4)$. What is $\log_a b$?

A $\dfrac{1}{4\pi}$

B $\dfrac{1}{\pi}$

C π

D 2π

E $10^{2\pi}$

Solution:

The circumference is 2π times the radius, so

$$\log_{10}(b^4) = 2\pi \log_{10}(a^2).$$

Rewriting, $4\log_{10} b = 4\pi \log_{10} a$, hence $\log_{10} b = \pi \log_{10} a$.

Therefore $\log_a b = \dfrac{\log_{10} b}{\log_{10} a} = \pi$.

Thus, the correct answer is **C**.

15. On each side of a unit square, an equilateral triangle of side length 1 is constructed. On each new side of each equilateral triangle, another equilateral triangle of side length 1 is constructed. The interiors of the square and the 12 triangles have no points in common. Let R be the region formed by the union of the square and all the triangles, and let S be the smallest convex polygon that contains R. What is the area of the region that is inside S but outside R?

A $\frac{1}{4}$

B $\frac{\sqrt{2}}{4}$

C 1

D $\sqrt{3}$

E $2\sqrt{3}$

Solution:

The convex hull S differs from R only near the four corners of the square, where a small triangular gap forms. Each gap triangle has two sides of length 1 (outer edges of adjacent triangles).

The angle between those two sides is $360^\circ - 90^\circ - 4 \cdot 60^\circ = 30^\circ$, so each gap has area

$$\tfrac{1}{2} \cdot 1 \cdot 1 \cdot \sin 30^\circ = \tfrac{1}{4}.$$

The total area is $4 \cdot \frac{1}{4} = 1$.

Thus, the correct answer is **C**.

16. A rectangular floor measures a feet by b feet, where a and b are positive integers with $b > a$. An artist paints a rectangle on the floor with the sides of the rectangle parallel to the sides of the floor. The unpainted part of the floor forms a border of width 1 foot around the painted rectangle and occupies half the area of the entire floor. How many possibilities are there for the ordered pair (a, b)?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

The painted rectangle measures $(a-2)$ by $(b-2)$ and has half the area of the floor, so

$$ab = 2(a-2)(b-2).$$

Expanding gives $0 = ab - 4a - 4b + 8$, and adding 8 yields $(a-4)(b-4) = 8$.

With $b > a > 0$, the only valid factor pairs of 8 are $(a-4, b-4) = (1, 8)$ and $(2, 4)$, giving $(a, b) = (5, 12)$ and $(6, 8)$.

There are 2 possibilities.

Thus, the correct answer is **B**.

17. Let A, B and C be three distinct points on the graph of $y = x^2$ such that line AB is parallel to the x-axis and $\triangle ABC$ is a right triangle with area 2008. What is the sum of the digits of the y-coordinate of C?

- A 16
- B 17
- C 18
- D 19
- E 20

Solution:

Since AB is horizontal, take $A = (a, a^2)$ and $B = (-a, a^2)$, and let $C = (c, c^2)$. The right angle cannot be at A or B (that would need $c = \pm a$), so it is at C.

Then $CA \perp CB$ gives $(c + a)(c - a) = -1$, so $a^2 - c^2 = 1$. This value is the height of the triangle above $\overline{AB}$.

The area is $\frac{1}{2} \cdot AB \cdot \text{height} = \frac{1}{2}(2|a|)(1) = |a| = 2008$, so $a^2 = 2008^2 = 4{,}032{,}064$ and the y-coordinate of C is $c^2 = a^2 - 1 = 4{,}032{,}063$.

Its digit sum is $4 + 0 + 3 + 2 + 0 + 6 + 3 = 18$.

Thus, the correct answer is **C**.

18. A pyramid has a square base $ABCD$ and vertex E. The area of square $ABCD$ is 196, and the areas of $\triangle ABE$ and $\triangle CDE$ are 105 and 91, respectively. What is the volume of the pyramid?

A 392

B $196\sqrt{6}$

C $392\sqrt{2}$

D $392\sqrt{3}$

E 784

Solution:

The square has side $\sqrt{196} = 14$. Let F and G be the feet of the perpendiculars from E to AB and CD. Then $FG = 14, EF = \frac{2\cdot105}{14} = 15,$ and $EG = \frac{2\cdot91}{14} = 13.$

Triangle EFG lies in a plane perpendicular to the base, so its altitude to FG is the pyramid's height. By Heron's formula with $s = 21,$ its area is $\sqrt{21 \cdot 6 \cdot 8 \cdot 7} = 84,$ so the altitude to FG is $\frac{2\cdot84}{14} = 12.$

The volume is $\frac{1}{3} \cdot 196 \cdot 12 = 784.$

Thus, the correct answer is **E**.

19. A function f is defined by $f(z) = (4+i)z^2 + \alpha z + \gamma$ for all complex numbers z, where α and γ are complex numbers and $i^2 = -1$. Suppose that $f(1)$ and $f(i)$ are both real. What is the smallest possible value of $|\alpha| + |\gamma|$?

A 1

B $\sqrt{2}$

C 2

D $2\sqrt{2}$

E 4

Solution:

Let $\alpha = a + bi$ and $\gamma = c + di$. Then $f(1) = (4 + a + c) + (1 + b + d)i$ and $f(i) = (-4 - b + c) + (-1 + a + d)i$.

Both being real forces $1 + b + d = 0$ and $-1 + a + d = 0$, i.e. $a = 1 - d$ and $b = -1 - d$.

Hence

$$|\alpha| + |\gamma| = \sqrt{(1-d)^2 + (1+d)^2} + \sqrt{c^2 + d^2} = \sqrt{2 + 2d^2} + \sqrt{c^2 + d^2},$$

which is smallest when $c = d = 0$, giving $\sqrt{2}$.

Thus, the correct answer is **B**.

20. Michael walks at the rate of 5 feet per second on a long straight path. Trash pails are located every 200 feet along the path. A garbage truck travels at 10 feet per second in the same direction as Michael and stops for 30 seconds at each pail. As Michael passes a pail, he notices the truck ahead of him just leaving the next pail. How many times will Michael and the truck meet?

- A 4
- B 5
- C 6
- D 7
- E 8

Solution:

Number the pails so Michael is at pail 0 and the truck at pail 1 at time $0.$ Michael reaches pail n at $40n$ seconds. The truck spends 20 seconds between pails and 30 stopped, so it leaves pail n at $50(n-1)$ seconds and (for $n \geq 2$) arrives at $50(n-1)-30.$

Michael is at pail n while the truck is there exactly when $50(n-1)-30 \leq 40n \leq 50(n-1),$ which simplifies to $5 \leq n \leq 8.$ So they meet at pail 5 (at $t=200,$ as the truck departs), pail 6 ($t=240$), pail 7 ($t=280$), and pail 8 ($t=320,$ as the truck arrives).

Between pails 6 and 7 the truck (moving at 10 ft/s) pulls ahead of and is then overtaken by Michael once more, adding one crossing. In all, they meet 5 times.

Thus, the correct answer is **B**.

21. Two circles of radius 1 are to be constructed as follows. The center of circle A is chosen uniformly and at random from the line segment joining $(0,0)$ to $(2,0)$. The center of circle B is chosen uniformly and at random, and independently of the first choice, from the line segment joining $(0,1)$ to $(2,1)$. What is the probability that circles A and B intersect?

A $\dfrac{2+\sqrt{2}}{4}$

B $\dfrac{3\sqrt{3}+2}{8}$

C $\dfrac{2\sqrt{2}-1}{2}$

D $\dfrac{2+\sqrt{3}}{4}$

E $\dfrac{4\sqrt{3}-3}{4}$

Solution:

Let the centers be $(a,0)$ and $(b,1)$ with $a,b \in [0,2]$. The circles (radius 1 each) intersect iff the distance between centers is at most 2 :

$$\sqrt{(a-b)^2+1} \leq 2 \iff |a-b| \leq \sqrt{3}.$$

The pairs (a,b) fill the square $[0,2]^2$ of area 4. The failing region $|a-b| > \sqrt{3}$ is two right triangles, each with legs $2-\sqrt{3}$, of total area $(2-\sqrt{3})^2 = 7-4\sqrt{3}.$

So the favorable area is $4-(7-4\sqrt{3}) = 4\sqrt{3}-3$, and the probability is

$$\frac{4\sqrt{3}-3}{4}.$$

Thus, the correct answer is **E**.

22. A parking lot has 16 spaces in a row. Twelve cars arrive, each of which requires one parking space, and their drivers choose their spaces at random from among the available spaces. Auntie Em then arrives in her SUV, which requires 2 adjacent spaces. What is the probability that she is able to park?

A $\frac{11}{20}$

B $\frac{4}{7}$

C $\frac{81}{140}$

D $\frac{3}{5}$

E $\frac{17}{28}$

Solution:

After the 12 cars park, 4 spaces are empty, equally likely to be any 4 of the $16,$ for $\binom{16}{4} = 1820$ equally likely sets.

Auntie Em fails exactly when no two empty spaces are adjacent. The number of ways to place 4 non-adjacent empties among 16 is $\binom{13}{4} = 715.$

So the probability she can park is

$$1 - \frac{715}{1820} = \frac{1105}{1820} = \frac{17}{28}.$$

Thus, the correct answer is **E**.

23. The sum of the base-10 logarithms of the divisors of 10^n is 792. What is n?

A 11

B 12

C 13

D 14

E 15

Solution:

The sum of the base-10 logs of the divisors is the log of their product. A number N with $d(N)$ divisors has divisor product $N^{d(N)/2}$.

Here $N = 10^n$ has $d(N) = (n+1)^2$ divisors, so the product is $(10^n)^{(n+1)^2/2}$ and its log is

$$\frac{n(n+1)^2}{2} = 792.$$

Thus $n(n+1)^2 = 1584 = 11 \cdot 144 = 11 \cdot 12^2$, giving $n = 11$.

Thus, the correct answer is **A**.

24. Let $A_0 = (0,0)$. Distinct points $A_1, A_2, \ldots$ lie on the x-axis, and distinct points $B_1, B_2, \ldots$ lie on the graph of $y = \sqrt{x}$. For every positive integer n, $A_{n-1}B_nA_n$ is an equilateral triangle. What is the least n for which the length $A_0A_n \geq 100$?

A 13

B 15

C 17

D 19

E 21

Solution:

Let $a_n = A_0A_n$ and $c_n = a_n - a_{n-1}$ be the base of the nth equilateral triangle. Its apex B_n lies above the midpoint at height $\frac{\sqrt{3}}{2}c_n$, and being on $y = \sqrt{x}$ gives

$$\left(\tfrac{\sqrt{3}}{2}c_n\right)^2 = a_{n-1} + \tfrac{c_n}{2}, \quad \text{i.e.} \quad \tfrac{3}{4}c_n^2 = a_{n-1} + \tfrac{c_n}{2}.$$

Writing the same relation for the previous triangle and subtracting gives $c_n = c_{n-1} + \frac{2}{3}$, and with $c_1 = \frac{2}{3}$ we get $c_n = \frac{2n}{3}$. Summing,

$$a_n = \sum_{k=1}^{n} \frac{2k}{3} = \frac{n(n+1)}{3}.$$

We need $\frac{n(n+1)}{3} \geq 100$, i.e. $n(n+1) \geq 300$. Since $16 \cdot 17 = 272$ and $17 \cdot 18 = 306$, the least such n is 17.

Thus, the correct answer is **C**.

25. Let $ABCD$ be a trapezoid with $AB \parallel CD, AB = 11, BC = 5, CD = 19,$ and $DA = 7.$ Bisectors of $\angle A$ and $\angle D$ meet at $P,$ and bisectors of $\angle B$ and $\angle C$ meet at $Q.$ What is the area of hexagon $ABQCDP$?

A $28\sqrt{3}$

B $30\sqrt{3}$

C $32\sqrt{3}$

D $35\sqrt{3}$

E $36\sqrt{3}$

Solution:

Because $AB \parallel CD, \angle A + \angle D = 180^\circ,$ so the bisectors of $\angle A$ and $\angle D$ meet at right angles, $\angle APD = 90^\circ.$ Then the midpoint M of $\overline{AD}$ is the circumcenter of right triangle $APD,$ giving $MP = MA = MD$ and $MP \parallel AB.$ The same holds for the midpoint N of $\overline{BC}$ with $Q,$ so M, P, Q, N are collinear on the midline.

The midline has length $\frac{AB+CD}{2} = 15,$ while $MP = \frac{AD}{2} = \frac{7}{2}$ and $QN = \frac{BC}{2} = \frac{5}{2}.$ Hence $PQ = 15 - \frac{7}{2} - \frac{5}{2} = 9.$

Drawing $AE \parallel BC$ with E on $\overline{CD}$ gives $AE = 5$ and $DE = CD - AB = 8.$ In $\triangle ADE, \cos(\angle AED) = \frac{8^2+5^2-7^2}{2 \cdot 8 \cdot 5} = \frac{1}{2},$ so $\angle AED = 60^\circ$ and the trapezoid's height is $AF = 5\sin 60^\circ = \frac{5\sqrt{3}}{2}.$

The segment PQ sits at half the height, so the hexagon splits into two trapezoids and

$$[ABQCDP] = \frac{AF}{4}\big(AB + CD + 2\,PQ\big) = \frac{5\sqrt{3}/2}{4}(11 + 19 + 18) = 30\sqrt{3}.$$

Thus, the correct answer is **B**.

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