2008 AMC 12A Solutions

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1. A bakery owner turns on his doughnut machine at $8:30$ am. At $11:10$ am the machine has completed one third of the day's job. At what time will the doughnut machine complete the job?

A $1:50$ pm

B $3:00$ pm

C $3:30$ pm

D $4:30$ pm

E $5:50$ pm

Solution:

From $8:30$ am to $11:10$ am is 2 hours 40 minutes, or 160 minutes, to complete one third of the job.

The whole job then takes $3 \cdot 160 = 480$ minutes, or 8 hours. Adding 8 hours to $8:30$ am gives $4:30$ pm.

Thus, **D** is the correct answer.

2. What is the reciprocal of

$$\frac{1}{2}+\frac{2}{3}?$$

A $\frac{6}{7}$

B $\frac{7}{6}$

C $\frac{5}{3}$

D 3

E $\frac{7}{2}$

Solution:

Using a common denominator,

$$\frac{1}{2}+\frac{2}{3}=\frac{3}{6}+\frac{4}{6}=\frac{7}{6}.$$

The reciprocal of $\frac{7}{6}$ is $\frac{6}{7}$.

Thus, **A** is the correct answer.

3. Suppose that $\frac{2}{3}$ of 10 bananas are worth as much as 8 oranges. How many oranges are worth as much as $\frac{1}{2}$ of 5 bananas?

- A 2
- B $\frac{5}{2}$
- C 3
- D $\frac{7}{2}$
- E 4

Solution:

Since $\frac{2}{3}$ of 10 bananas is $\frac{20}{3}$ bananas, worth 8 oranges, one banana is worth

$$8 \div \frac{20}{3} = \frac{24}{20} = \frac{6}{5}$$

oranges.

Then $\frac{1}{2}$ of 5 bananas is $\frac{5}{2}$ bananas, worth

$$\frac{5}{2} \cdot \frac{6}{5} = 3$$

oranges.

Thus, **C** is the correct answer.

4. Which of the following is equal to the product

$$\frac{8}{4} \cdot \frac{12}{8} \cdot \frac{16}{12} \cdots \frac{4n+4}{4n} \cdots \frac{2008}{2004}?$$

- A 251
- B 502
- C 1004
- D 2008
- E 4016

Solution:

Every denominator except the first cancels with the numerator of the preceding fraction, so the product collapses to

$$\frac{2008}{4} = 502.$$

Thus, **B** is the correct answer.

5. Suppose that

$$\frac{2x}{3} - \frac{x}{6}$$

is an integer. Which of the following statements must be true about x?

- A It is negative.
- B It is even, but not necessarily a multiple of 3.
- C It is a multiple of 3, but not necessarily even.
- D It is a multiple of 6, but not necessarily a multiple of 12.
- E It is a multiple of 12.

Solution:

Combining the fractions,

$$\frac{2x}{3} - \frac{x}{6} = \frac{4x - x}{6} = \frac{x}{2}.$$

This is an integer exactly when x is even. The example $x = 4$ is even but not a multiple of 3, which rules out every other statement.

Thus, **B** is the correct answer.

6. Heather compares the price of a new computer at two different stores. Store A offers 15% off the sticker price followed by a \$90 rebate, and store B offers 25% off the same sticker price with no rebate. Heather saves \$15 by buying the computer at store A instead of store B. What is the sticker price of the computer, in dollars?

A 750

B 900

C 1000

D 1050

E 1500

Solution:

Let x be the sticker price in dollars. Store A charges $0.85x - 90$ dollars, and store B charges $0.75x$ dollars.

Since store A is 15 dollars cheaper,

$$0.85x - 90 = 0.75x - 15,$$

so $0.10x = 75$ and $x = 750.$

Thus, **A** is the correct answer.

7. While Steve and LeRoy are fishing 1 mile from shore, their boat springs a leak, and water comes in at a constant rate of 10 gallons per minute. The boat will sink if it takes in more than 30 gallons of water. Steve starts rowing toward the shore at a constant rate of 4 miles per hour while LeRoy bails water out of the boat. What is the slowest rate, in gallons per minute, at which LeRoy can bail if they are to reach the shore without sinking?

- A 2
- B 4
- C 6
- D 8
- E 10

Solution:

Rowing 1 mile at 4 miles per hour takes $\frac{1}{4}$ hour, or 15 minutes. In that time $15 \cdot 10 = 150$ gallons of water enter the boat.

Since at most 30 gallons may remain, LeRoy must bail $150 - 30 = 120$ gallons in 15 minutes, a rate of

$$\frac{120}{15} = 8$$

gallons per minute.

Thus, **D** is the correct answer.

8. What is the volume of a cube whose surface area is twice that of a cube with volume 1?

A $\sqrt{2}$

B 2

C $2\sqrt{2}$

D 4

E 8

Solution:

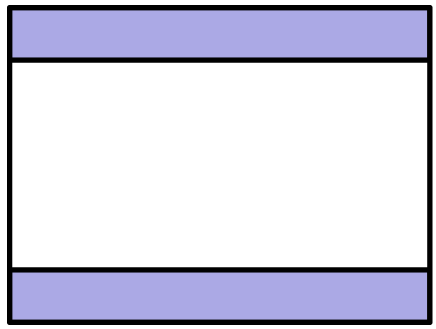
The cube with volume 1 has side 1 and surface area 6. The larger cube has surface area 12, so if its side is s, then $6s^2 = 12$, giving $s = \sqrt{2}$.

Its volume is

$$(\sqrt{2})^3 = 2\sqrt{2}.$$

Thus, **C** is the correct answer.

9. Older television screens have an aspect ratio of $4:3$. That is, the ratio of the width to the height is $4:3$. The aspect ratio of many movies is not $4:3$, so they are sometimes shown on a television screen by "letterboxing" — darkening strips of equal height at the top and bottom of the screen, as shown. Suppose a movie has an aspect ratio of $2:1$ and is shown on an older television screen with a 27-inch diagonal. What is the height, in inches, of each darkened strip?



- A 2
- B 2.25
- C 2.5
- D 2.7
- E 3

Solution:

Since the sides and diagonal are in ratio $3:4:5$, the height is $\frac{3}{5} \cdot 27 = 16.2$ inches and the width is $\frac{4}{5} \cdot 27 = 21.6$ inches.

The movie has aspect ratio $2:1$, so its height is $\frac{21.6}{2} = 10.8$ inches.

Each darkened strip therefore has height

$$\frac{16.2 - 10.8}{2} = 2.7$$

inches.

Thus, **D** is the correct answer.

10. Doug can paint a room in 5 hours. Dave can paint the same room in 7 hours. Doug and Dave paint the room together and take a one-hour break for lunch. Let t be the total time, in hours, required for them to complete the job working together, including lunch. Which of the following equations is satisfied by t?

A $\left(\frac{1}{5}+\frac{1}{7}\right)(t+1)=1$

B $\left(\frac{1}{5}+\frac{1}{7}\right)t+1=1$

C $\left(\frac{1}{5}+\frac{1}{7}\right)t=1$

D $\left(\frac{1}{5}+\frac{1}{7}\right)(t-1)=1$

E $(5+7)t=1$

Solution:

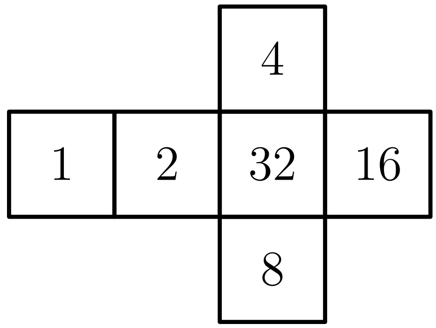
In one hour Doug paints $\frac{1}{5}$ of the room and Dave paints $\frac{1}{7}$, so together they paint $\frac{1}{5}+\frac{1}{7}$ of the room per hour.

Of the total time t, one hour is spent at lunch, so they work for $t-1$ hours. The fraction painted must equal 1, giving

$$\left(\frac{1}{5}+\frac{1}{7}\right)(t-1)=1.$$

Thus, **D** is the correct answer.

11. Three cubes are each formed from the pattern shown. They are then stacked on a table one on top of another so that the 13 visible numbers have the greatest possible sum. What is that sum?



A 154

B 159

C 164

D 167

E 189

Solution:

The six faces of each cube sum to $1+2+4+8+16+32=63$. From the pattern, the pairs of opposite faces are 1 & 32, 2 & 16, and 4 & 8.

Each of the two lower cubes hides a pair of opposite faces (top and bottom); hiding the pair $4+8=12$ is best. The top cube hides only its bottom face, so hide the 1.

The greatest sum is

$$3\cdot 63-2\cdot 12-1=189-24-1=164.$$

Thus, **C** is the correct answer.

12. A function f has domain $[0, 2]$ and range $[0, 1]$. (The notation $[a, b]$ denotes $\{x : a \leq x \leq b\}$.) What are the domain and range, respectively, of the function g defined by

$$g(x) = 1 - f(x + 1)?$$

A $[-1, 1], [-1, 0]$

B $[-1, 1], [0, 1]$

C $[0, 2], [-1, 0]$

D $[1, 3], [-1, 0]$

E $[1, 3], [0, 1]$

Solution:

The value $f(x+1)$ is defined when $0 \leq x + 1 \leq 2$, that is, $-1 \leq x \leq 1$, so the domain of g is $[-1, 1]$.

As $f(x+1)$ ranges over $[0, 1]$, the value $1 - f(x+1)$ ranges over $[0, 1]$ as well, so the range of g is $[0, 1]$.

Thus, **B** is the correct answer.

13. Points A and B lie on a circle centered at O, and $\angle AOB = 60^\circ$. A second circle is internally tangent to the first and tangent to both OA and OB. What is the ratio of the area of the smaller circle to that of the larger circle?

A $\frac{1}{16}$

B $\frac{1}{9}$

C $\frac{1}{8}$

D $\frac{1}{6}$

E $\frac{1}{4}$

Solution:

Let r and R be the radii of the smaller and larger circles, and let E be the center of the smaller circle. By symmetry E lies on the bisector of $\angle AOB$, so OE makes a 30° angle with OA.

Dropping the radius ED perpendicular to OA gives a 30-60-90 triangle with $OE = 2 \cdot ED = 2r$. Since the circles are internally tangent, $OE = R - r$.

Then $R - r = 2r$, so $R = 3r$ and $\frac{r}{R} = \frac{1}{3}$. The ratio of areas is

$$\left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

Thus, **B** is the correct answer.

14. What is the area of the region defined by the inequality

$$|3x-18|+|2y+7|\leq 3?$$

A 3

B $\frac{7}{2}$

C 4

D $\frac{9}{2}$

E 5

Solution:

The region is a rhombus centered at $\left(6,-\frac{7}{2}\right)$. Setting $2y+7=0$ gives $|3x-18|\leq 3$, so $x\in[5,7]$, a horizontal diagonal of length 2.

Setting $3x-18=0$ gives $|2y+7|\leq 3$, so $y\in[-5,-2]$, a vertical diagonal of length 3.

The area of the rhombus is half the product of its diagonals,

$$\frac{1}{2}\cdot 2\cdot 3=3.$$

Thus, **A** is the correct answer.

15. Let $k = 2008^2 + 2^{2008}$. What is the units digit of $k^2 + 2^k$?

A 0

B 2

C 4

D 6

E 8

Solution:

The units digit of 2008^2 is 4. Since 2008 is a multiple of 4, the units digit of 2^{2008} is 6. Thus k has units digit 0, and so does k^2.

Both 2008^2 and 2^{2008} are multiples of 4, so k is a multiple of 4. Therefore the units digit of 2^k is 6.

The units digit of $k^2 + 2^k$ is then $0 + 6 = 6$.

Thus, **D** is the correct answer.

16. The numbers $\log(a^3b^7)$, $\log(a^5b^{12})$, and $\log(a^8b^{15})$ are the first three terms of an arithmetic sequence, and the 12th term of the sequence is $\log(b^n)$. What is n?

- A 40
- B 56
- C 76
- D 112
- E 143

Solution:

The three terms are $3\log a + 7\log b$, $5\log a + 12\log b$, and $8\log a + 15\log b$. Setting the two consecutive differences equal,

$$2\log a + 5\log b = 3\log a + 3\log b,$$

so $\log a = 2\log b$.

The first term is then $(3\cdot 2 + 7)\log b = 13\log b$, and the common difference is $(2\cdot 2 + 5)\log b = 9\log b$.

The 12th term is

$$(13 + 11\cdot 9)\log b = 112\log b = \log(b^{112}),$$

so $n = 112$.

Thus, **D** is the correct answer.

17. Let $a_1, a_2, \ldots$ be a sequence of integers determined by the rule $a_n = a_{n-1}/2$ if a_{n-1} is even and $a_n = 3a_{n-1} + 1$ if a_{n-1} is odd. For how many positive integers $a_1 \leq 2008$ is it true that a_1 is less than each of $a_2, a_3,$ and a_4?

A 250

B 251

C 501

D 502

E 1004

Solution:

If a_1 is even, then $a_2 = a_1/2 < a_1$, so the condition fails.

If $a_1 \equiv 1 \pmod 4$, then $a_2 = 3a_1 + 1$ is a multiple of 4, so $a_3 = (3a_1 + 1)/2$ and $a_4 = (3a_1 + 1)/4 \leq a_1$, and again the condition fails.

If $a_1 \equiv 3 \pmod 4$, then a_2 is even but not a multiple of 4, so $a_3 = (3a_1 + 1)/2 > a_1$, and a_3 is odd, giving $a_4 = 3a_3 + 1 > a_3 > a_1$. The condition holds.

Exactly $\frac{2008}{4} = 502$ values of $a_1 \leq 2008$ satisfy $a_1 \equiv 3 \pmod 4$.

Thus, **D** is the correct answer.

18. Triangle ABC, with sides of length $5, 6,$ and 7, has one vertex on the positive x-axis, one on the positive y-axis, and one on the positive z-axis. Let O be the origin. What is the volume of tetrahedron $OABC$?

A $\sqrt{85}$

B $\sqrt{90}$

C $\sqrt{95}$

D 10

E $\sqrt{105}$

Solution:

Let $A = (a, 0, 0), B = (0, b, 0), C = (0, 0, c)$. Assigning the sides,

$$a^2 + b^2 = 25, \quad b^2 + c^2 = 36, \quad a^2 + c^2 = 49.$$

Adding gives $a^2 + b^2 + c^2 = 55$, so $a^2 = 19, b^2 = 6$, and $c^2 = 30.$

The volume is

$$\frac{1}{6}abc = \frac{1}{6}\sqrt{19 \cdot 6 \cdot 30} = \frac{1}{6}\sqrt{3420} = \sqrt{95}.$$

Thus, **C** is the correct answer.

19. In the expansion of

$$\left(1+x+x^2+\cdots+x^{27}\right)\left(1+x+x^2+\cdots+x^{14}\right)^2,$$

what is the coefficient of x^{28}?

A 195

B 196

C 224

D 378

E 405

Solution:

Each term is x^{a+b+c} with $0 \le a \le 27$ and $0 \le b, c \le 14$. To get x^{28} we need $a = 28 - b - c$.

There are $(14+1)^2 = 225$ choices for (b, c). For every choice except $(b, c) = (0, 0)$, the required $a = 28 - b - c$ lies in $[0, 27]$, giving a valid term.

The coefficient of x^{28} is therefore $225 - 1 = 224$.

Thus, **C** is the correct answer.

20. Triangle ABC has $AC = 3, BC = 4$, and $AB = 5$. Point D is on AB, and CD bisects the right angle. The inscribed circles of $\triangle ADC$ and $\triangle BCD$ have radii r_a and r_b, respectively. What is r_a/r_b?

A $\frac{1}{28}(10 - \sqrt{2})$

B $\frac{3}{56}(10 - \sqrt{2})$

C $\frac{1}{14}(10 - \sqrt{2})$

D $\frac{5}{56}(10 - \sqrt{2})$

E $\frac{3}{28}(10 - \sqrt{2})$

Solution:

By the Angle Bisector Theorem, $AD : DB = CA : CB = 3 : 4$, so $AD = \frac{15}{7}$ and $BD = \frac{20}{7}$. The areas of $\triangle ADC$ and $\triangle BCD$ share base CD, so they are in ratio $3 : 4$, namely $\frac{18}{7}$ and $\frac{24}{7}$.

Splitting $\triangle ABC$ along CD, which meets each leg at 45°, gives

$$\frac{3 \cdot CD}{2\sqrt{2}} + \frac{4 \cdot CD}{2\sqrt{2}} = 6,$$

so $CD = \frac{12\sqrt{2}}{7}$.

Using $r = \text{area}/s$ with s the semiperimeter,

$$\frac{r_a}{r_b} = \frac{[ADC]}{[BCD]} \cdot \frac{s_b}{s_a} = \frac{3}{4} \cdot \frac{4 + \sqrt{2}}{3 + \sqrt{2}},$$

where the messy CD terms cancel after factoring.

Rationalizing, $\dfrac{4+\sqrt{2}}{3+\sqrt{2}} = \dfrac{10-\sqrt{2}}{7}$, so

$$\frac{r_a}{r_b} = \frac{3}{4} \cdot \frac{10-\sqrt{2}}{7} = \frac{3}{28}(10-\sqrt{2}).$$

Thus, **E** is the correct answer.

21. A permutation $(a_1, a_2, a_3, a_4, a_5)$ of $(1, 2, 3, 4, 5)$ is *heavy-tailed* if $a_1 + a_2 < a_4 + a_5$. What is the number of heavy-tailed permutations?

A 36

B 40

C 44

D 48

E 52

Solution:

Call a permutation balanced if $a_1 + a_2 = a_4 + a_5$. Reversing the entries swaps the two strict cases, so heavy-tailed and heavy-headed permutations are equally numerous.

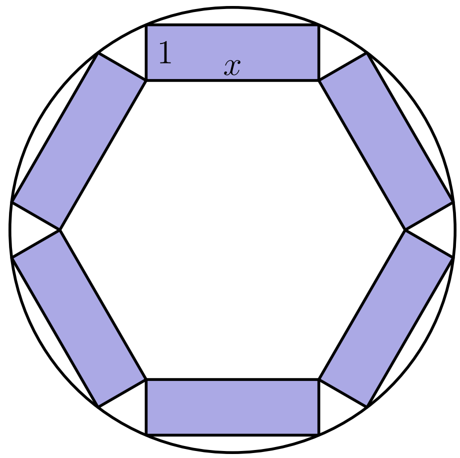
The total $1+2+3+4+5 = 15$ is odd, so in a balanced permutation a_3 must be odd, one of $1, 3, 5$. For each choice, the remaining four numbers split uniquely into two equal-sum pairs.

Any of the four can be a_1 (fixing a_2), and either remaining number can be a_4 (fixing a_5), giving $3 \cdot 4 \cdot 2 = 24$ balanced permutations.

The other $120 - 24 = 96$ permutations split evenly, so there are $\frac{96}{2} = 48$ heavy-tailed permutations.

Thus, **D** is the correct answer.

22. A round table has radius 4. Six rectangular place mats are placed on the table. Each place mat has width 1 and length x as shown. They are positioned so that each mat has two corners on the edge of the table, these two corners being endpoints of the same side of length x. Further, the mats are positioned so that the inner corners each touch an inner corner of an adjacent mat. What is x?



A $2\sqrt{5}-\sqrt{3}$

B 3

C $\dfrac{3\sqrt{7}-\sqrt{3}}{2}$

D $2\sqrt{3}$

E $\dfrac{5+2\sqrt{3}}{2}$

Solution:

Take one mat with outer corners P and Q, and let R be the point of the table's edge diametrically opposite P. Then $PR = 8$ is a diameter, so $\triangle PQR$ has a right angle at Q, with $PQ = x$.

Along QR, the inner corners of neighboring mats meet in an isosceles triangle with two sides of length x and vertex angle 120°, whose base is $\sqrt{3}\,x$. Hence $QR = \sqrt{3}\,x + 2$.

The Pythagorean Theorem gives

$$x^2 + \left(\sqrt{3}\,x + 2\right)^2 = 64,$$

which simplifies to $x^2 + \sqrt{3}\,x - 15 = 0.$

Taking the positive root,

$$x = \frac{-\sqrt{3} + \sqrt{63}}{2} = \frac{3\sqrt{7} - \sqrt{3}}{2}.$$

Thus, **C** is the correct answer.

23. The solutions of the equation

$$z^4 + 4z^3i - 6z^2 - 4zi - i = 0$$

are the vertices of a convex polygon in the complex plane. What is the area of the polygon?

A $2^{5/8}$

B $2^{3/4}$

C 2

D $2^{5/4}$

E $2^{3/2}$

Solution:

Adding $1 + i$ to both sides, the left side becomes

$$z^4 + 4z^3i - 6z^2 - 4zi + 1 = (z + i)^4,$$

so $(z + i)^4 = 1 + i.$

The four solutions for $w = z + i$ are equally spaced on a circle of radius $|1 + i|^{1/4} = (2^{1/2})^{1/4} = 2^{1/8}$, and they form a square. Subtracting i merely translates it.

A square inscribed in a circle of radius $2^{1/8}$ has diagonal $2 \cdot 2^{1/8} = 2^{9/8}$, so its side is $\frac{2^{9/8}}{\sqrt{2}} = 2^{5/8}.$

The area is

$$\left(2^{5/8}\right)^2 = 2^{5/4}.$$

Thus, **D** is the correct answer.

24. Triangle ABC has $\angle C = 60^\circ$ and $BC = 4$. Point D is the midpoint of BC. What is the largest possible value of $\tan(\angle BAD)$?

- A $\dfrac{\sqrt{3}}{6}$
- B $\dfrac{\sqrt{3}}{3}$
- C $\dfrac{\sqrt{3}}{2\sqrt{2}}$
- D $\dfrac{\sqrt{3}}{4\sqrt{2}-3}$
- E 1

Solution:

Place $C = (0,0), B = (2, 2\sqrt{3})$ so that $\angle C = 60^\circ$ and $BC = 4$, and let $A = (x, 0)$ with $x > 0$. Then $D = (1, \sqrt{3})$ is the midpoint of BC.

The lines AD and AB have slopes $\frac{\sqrt{3}}{1-x}$ and $\frac{2\sqrt{3}}{2-x}$. Using the tangent-difference formula and simplifying,

$$\tan(\angle BAD) = \frac{\sqrt{3}\,x}{x^2 - 3x + 8}.$$

Setting the derivative to zero gives $x^2 = 8$, so $x = 2\sqrt{2}$. Substituting,

$$\tan(\angle BAD) = \frac{2\sqrt{6}}{16 - 6\sqrt{2}} = \frac{\sqrt{6}}{8 - 3\sqrt{2}} = \frac{\sqrt{3}}{4\sqrt{2} - 3}.$$

Thus, **D** is the correct answer.

25. A sequence $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots$ of points in the coordinate plane satisfies

$$(a_{n+1}, b_{n+1}) = \left(\sqrt{3}\, a_n - b_n,\ \sqrt{3}\, b_n + a_n\right) \text{ for } n = 1, 2, 3, \ldots$$

Suppose that $(a_{100}, b_{100}) = (2, 4)$. What is $a_1 + b_1$?

A $-\frac{1}{2^{97}}$

B $-\frac{1}{2^{99}}$

C 0

D $\frac{1}{2^{98}}$

E $\frac{1}{2^{96}}$

Solution:

Let $z_n = a_n + b_n i$. Then

$$z_{n+1} = (\sqrt{3}\, a_n - b_n) + (\sqrt{3}\, b_n + a_n)i = (a_n + b_n i)(\sqrt{3} + i),$$

so $z_{n+1} = z_n(\sqrt{3} + i)$ and $z_{100} = z_1(\sqrt{3} + i)^{99}$.

Since $\sqrt{3} + i = 2(\cos 30^\circ + i \sin 30^\circ)$, De Moivre's theorem gives $(\sqrt{3} + i)^{99} = 2^{99}(\cos 2970^\circ + i \sin 2970^\circ)$. As 2970° is coterminal with 90°, this equals $2^{99} i$.

Thus $2 + 4i = z_1 \cdot 2^{99} i$, so

$$z_1 = \frac{2 + 4i}{2^{99} i} = \frac{4 - 2i}{2^{99}}.$$

Then $a_1 = \frac{4}{2^{99}}$ and $b_1 = -\frac{2}{2^{99}}$, so

$$a_1 + b_1 = \frac{2}{2^{99}} = \frac{1}{2^{98}}.$$

Thus, **D** is the correct answer.

Problems: https://live.poshenloh.com/past-contests/amc12/2008A