2007 AMC 12B Solutions

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1. Isabella's house has 3 bedrooms. Each bedroom is 12 feet long, 10 feet wide, and 8 feet high. Isabella must paint the walls of all the bedrooms. Doorways and windows, which will not be painted, occupy 60 square feet in each bedroom. How many square feet of walls must be painted?

   A 678

   B 768

   C 786

   D 867

   E 876

Solution:

The perimeter of each bedroom floor is $2(12+10)=44$ feet.

So the wall area in one bedroom is $44 \cdot 8-60=352-60=292$ square feet. Across all three bedrooms, Isabella paints $3 \cdot 292=876$ square feet.

Thus, the correct answer is **E**.

2. A college student drove his compact car 120 miles home for the weekend and averaged 30 miles per gallon. On the return trip the student drove his parents' SUV and averaged only 20 miles per gallon. What was the average gas mileage, in miles per gallon, for the round trip?

- A 22
- B 24
- C 25
- D 26
- E 28

Solution:

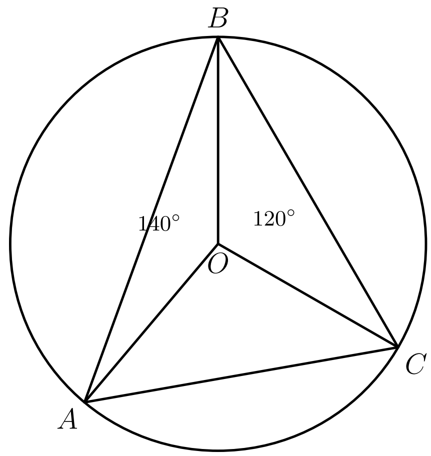
The trip home uses $120/30 = 4$ gallons, and the return trip uses $120/20 = 6$ gallons.

The round trip covers 240 miles on 10 gallons, so the average is

$$\frac{240}{10} = 24 \text{ miles per gallon.}$$

Thus, the correct answer is **B**.

3. The point O is the center of the circle circumscribed about $\triangle ABC$, with $\angle BOC = 120^\circ$ and $\angle AOB = 140^\circ$, as shown. What is the degree measure of $\angle ABC$?



A 35

B 40

C 45

D 50

E 60

Solution:

The angles around O sum to 360°, so

$$\angle AOC = 360^\circ - 140^\circ - 120^\circ = 100^\circ.$$

By the inscribed angle theorem, $\angle ABC$ subtends the same arc AC as the central angle $\angle AOC$, so

$$\angle ABC = \frac{1}{2}\angle AOC = 50^\circ.$$

Thus, the correct answer is **D**.

4. At Frank's Fruit Market, 3 bananas cost as much as 2 apples, and 6 apples cost as much as 4 oranges. How many oranges cost as much as 18 bananas?

- A 6
- B 8
- C 9
- D 12
- E 18

Solution:

Since 3 bananas cost as much as 2 apples, 18 bananas cost as much as 12 apples.

Since 6 apples cost as much as 4 oranges, 12 apples cost as much as 8 oranges. Therefore 18 bananas cost as much as 8 oranges.

Thus, the correct answer is **B**.

5. The 2007 AMC 12 contests will be scored by awarding 6 points for each correct response, 0 points for each incorrect response, and 1.5 points for each problem left unanswered. After looking over the 25 problems, Sarah has decided to attempt the first 22 and leave the last 3 unanswered. How many of the first 22 problems must she solve correctly in order to score at least 100 points?

- A 13
- B 14
- C 15
- D 16
- E 17

Solution:

The three unanswered problems give $3 \cdot 1.5 = 4.5$ points.

So Sarah needs at least $100 - 4.5 = 95.5$ points from correct answers. Since

$$15 < \frac{95.5}{6} < 16,$$

she must solve at least 16 problems correctly, which would give her $16 \cdot 6 + 4.5 = 100.5$ points.

Thus, the correct answer is **D**.

6. Triangle ABC has side lengths $AB = 5, BC = 6,$ and $AC = 7.$ Two bugs start simultaneously from A and crawl along the sides of the triangle in opposite directions at the same speed. They meet at point $D.$ What is BD?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

The perimeter is $5 + 6 + 7 = 18,$ so each bug crawls 9 before they meet.

The bug going $A \to B \to C$ reaches D on side $BC,$ having traveled $AB + BD = 9.$ Since $AB = 5,$ we get $BD = 4.$

Thus, the correct answer is **D**.

7. All sides of the convex pentagon $ABCDE$ are of equal length, and $\angle A = \angle B = 90^\circ$. What is the degree measure of $\angle E$?

A 90

B 108

C 120

D 144

E 150

Solution:

Because $AB = BC = EA$ and $\angle A = \angle B = 90^\circ$, quadrilateral $ABCE$ is a square, so $\angle AEC = 90^\circ$ and EC equals the common side length.

Then $CD = DE = EC$, so $\triangle CDE$ is equilateral and $\angle CED = 60^\circ$. Therefore

$$\angle E = \angle AEC + \angle CED = 90^\circ + 60^\circ = 150^\circ.$$

Thus, the correct answer is **E**.

8. Tom's age is T years, which is also the sum of the ages of his three children. His age N years ago was twice the sum of their ages then. What is T/N?

- A 2
- B 3
- C 4
- D 5
- E 6

Solution:

Tom's age N years ago was $T - N$. His three children were each N years younger, so their ages then totaled $T - 3N$.

The condition gives

$$T - N = 2(T - 3N),$$

so $5N = T$ and $T/N = 5$.

Thus, the correct answer is **D**.

9. A function f has the property that $f(3x-1)=x^2+x+1$ for all real numbers x. What is $f(5)$?

A 7

B 13

C 31

D 111

E 211

Solution:

Setting $3x-1=5$ gives $x=2$.

Then

$$f(5)=2^2+2+1=7.$$

Thus, the correct answer is **A**.

10. Some boys and girls are having a car wash to raise money for a class trip to China. Initially 40% of the group are girls. Shortly thereafter two girls leave and two boys arrive, and then 30% of the group are girls. How many girls were initially in the group?

- A 4
- B 6
- C 8
- D 10
- E 12

Solution:

Since two girls leave while two boys arrive, the total group size is unchanged. The drop from 40% to 30% girls corresponds to the two girls who left.

So those two girls are 10% of the group, meaning the group has 20 people. The initial number of girls was 40% of 20, or 8.

Thus, the correct answer is **C**.

11. The angles of quadrilateral $ABCD$ satisfy $\angle A = 2\angle B = 3\angle C = 4\angle D$. What is the degree measure of $\angle A$, rounded to the nearest whole number?

A 125

B 144

C 153

D 173

E 180

Solution:

Let $x = \angle A$. Then $\angle B = \frac{x}{2}$, $\angle C = \frac{x}{3}$, and $\angle D = \frac{x}{4}$.

The angle sum gives

$$x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{25x}{12} = 360,$$

so $x = \dfrac{12 \cdot 360}{25} = 172.8 \approx 173.$

Thus, the correct answer is **D**.

12. A teacher gave a test to a class in which 10% of the students are juniors and 90% are seniors. The average score on the test was $84.$ The juniors all received the same score, and the average score of the seniors was $83.$ What score did each of the juniors receive on the test?

A 85

B 88

C 93

D 94

E 98

Solution:

Take a class of 10 students, so there is 1 junior and 9 seniors.

The total of all scores is $10 \cdot 84 = 840,$ and the seniors contribute $9 \cdot 83 = 747.$ So the junior scored

$$840 - 747 = 93.$$

Thus, the correct answer is **C**.

13. A traffic light runs repeatedly through the following cycle: green for 30 seconds, then yellow for 3 seconds, and then red for 30 seconds. Leah picks a random three-second time interval to watch the light. What is the probability that the color changes while she is watching?

A $\frac{1}{63}$

B $\frac{1}{21}$

C $\frac{1}{10}$

D $\frac{1}{7}$

E $\frac{1}{3}$

Solution:

The cycle length is $30 + 3 + 30 = 63$ seconds, with three color changes per cycle.

Leah sees a change exactly when her three-second interval starts within the 3 seconds before a switch. That gives $3 \cdot 3 = 9$ favorable seconds out of 63, a probability of

$$\frac{9}{63} = \frac{1}{7}.$$

Thus, the correct answer is **D**.

14. Point P is inside equilateral $\triangle ABC$. Points $Q, R,$ and S are the feet of the perpendiculars from P to $\overline{AB}, \overline{BC},$ and $\overline{CA},$ respectively. Given that $PQ = 1, PR = 2,$ and $PS = 3,$ what is AB?

A 4

B $3\sqrt{3}$

C 6

D $4\sqrt{3}$

E 9

Solution:

Let $s = AB.$ Joining P to the vertices splits the triangle into $\triangle PAB, \triangle PBC,$ and $\triangle PCA,$ with areas $\frac{s}{2}, s,$ and $\frac{3s}{2}.$

Their total is $3s,$ which must equal the area $\frac{\sqrt{3}}{4}s^2$ of the equilateral triangle. So

$$3s = \frac{\sqrt{3}}{4}s^2,$$

giving $s = \dfrac{12}{\sqrt{3}} = 4\sqrt{3}.$

Thus, the correct answer is **D**.

15. The geometric series $a + ar + ar^2 + \cdots$ has a sum of 7, and the terms involving odd powers of r have a sum of 3. What is $a + r$?

A $\frac{4}{3}$

B $\frac{12}{7}$

C $\frac{3}{2}$

D $\frac{7}{3}$

E $\frac{5}{2}$

Solution:

The odd-power terms are $ar + ar^3 + \cdots = r(a + ar^2 + \cdots)$, that is, r times the even-power terms. The even-power terms sum to $7 - 3 = 4$.

So $3 = 4r$, giving $r = \frac{3}{4}$. Then $a = 7(1 - r) = \frac{7}{4}$, and

$$a + r = \frac{7}{4} + \frac{3}{4} = \frac{5}{2}.$$

Thus, the correct answer is **E**.

16. Each face of a regular tetrahedron is painted either red, white, or blue. Two colorings are considered indistinguishable if two congruent tetrahedra with those colorings can be rotated so that their appearances are identical. How many distinguishable colorings are possible?

A 15

B 18

C 27

D 54

E 81

Solution:

The rotation group of the tetrahedron has 12 elements: the identity, 8 rotations of order 3 about a vertex-face axis, and 3 rotations of order 2 about an edge-midpoint axis.

The identity fixes all $3^4 = 81$ colorings. Each vertex rotation fixes one face and cycles the other three, so it fixes $3^2 = 9$ colorings; likewise each edge rotation swaps two pairs of faces and fixes $3^2 = 9.$

By Burnside's lemma the number of distinguishable colorings is

$$\frac{81 + 8 \cdot 9 + 3 \cdot 9}{12} = \frac{180}{12} = 15.$$

Thus, the correct answer is **A**.

17. If a is a nonzero integer and b is a positive number such that $ab^2 = \log_{10} b$, what is the median of the set $\{0, 1, a, b, 1/b\}$?

A 0

B 1

C a

D b

E $\frac{1}{b}$

Solution:

Because $b < 10^b$ for all $b > 0$, it follows that $\log_{10} b < b$. If $b \geq 1$, then $0 < \frac{\log_{10} b}{b^2} < 1$, so a could not be a nonzero integer.

Hence $0 < b < 1$, so $\log_{10} b < 0$ and $a = \frac{\log_{10} b}{b^2} < 0$. Thus $a < 0 < b < 1 < \frac{1}{b}$, and the middle value of the sorted set is b.

Thus, the correct answer is **D**.

18. Let $a, b,$ and c be digits with $a \neq 0.$ The three-digit integer $\overline{abc}$ lies one third of the way from the square of a positive integer to the square of the next larger integer. The integer $\overline{acb}$ lies two thirds of the way between the same two squares. What is $a + b + c$?

- A 10
- B 13
- C 16
- D 18
- E 21

Solution:

Let the smaller square be N^2, so the larger is $(N+1)^2$ and the gap is $2N+1.$ Then

$$\overline{abc} = N^2 + \frac{2N+1}{3}, \qquad \overline{acb} = N^2 + \frac{2(2N+1)}{3}.$$

Subtracting, $\overline{acb} - \overline{abc} = 9(c-b) = \dfrac{2N+1}{3}$, so $27(c-b) = 2N+1.$ If $c-b=0$ or $2,$ then N is not an integer; if $c-b \geq 3,$ then $N \geq 40$ and the numbers are not three digits.

So $c-b=1,$ giving $N=13.$ The points one third and two thirds of the way from $13^2 = 169$ to $14^2 = 196$ are 178 and $187,$ so $a+b+c = 1+7+8 = 16.$

Thus, the correct answer is **C**.

19. Rhombus $ABCD$, with side length 6, is rolled to form a cylinder of volume 6 by taping $\overline{AB}$ to $\overline{DC}$. What is $\sin(\angle ABC)$?

A $\frac{\pi}{9}$

B $\frac{1}{2}$

C $\frac{\pi}{6}$

D $\frac{\pi}{4}$

E $\frac{\sqrt{3}}{2}$

Solution:

Let $\theta = \angle ABC$. The base circle has circumference 6, so its radius is $\frac{6}{2\pi} = \frac{3}{\pi}$. The height of the cylinder is the rhombus altitude $6\sin\theta$.

The volume is

$$\pi\left(\frac{3}{\pi}\right)^2(6\sin\theta) = \frac{54}{\pi}\sin\theta = 6,$$

so $\sin\theta = \frac{\pi}{9}$.

Thus, the correct answer is **A**.

20. The parallelogram bounded by the lines $y = ax + c, y = ax + d, y = bx + c$, and $y = bx + d$ has area 18. The parallelogram bounded by the lines $y = ax + c, y = ax - d, y = bx + c$, and $y = bx - d$ has area 72. Given that a, b, c, and d are positive integers, what is the smallest possible value of $a + b + c + d$?

A 13

B 14

C 15

D 16

E 17

Solution:

Two vertices of the first parallelogram lie at $(0, c)$ and $(0, d)$, and the other two have x-coordinates $\pm\frac{c-d}{b-a}$. Its area works out to $\frac{(c-d)^2}{|b-a|} = 18$. The same computation for the second gives $\frac{(c+d)^2}{|b-a|} = 72$.

So $(c-d)^2 = 18|b-a|$ and $(c+d)^2 = 72|b-a|$. Subtracting, $4cd = 54|b-a|$, i.e. $2cd = 27|b-a|$.

Thus $|b-a|$ is even, so $a+b$ is smallest with $\{a, b\} = \{1, 3\}$; and cd is a multiple of 27, so $c+d$ is smallest with $\{c, d\} = \{3, 9\}$. These satisfy all conditions, giving $a + b + c + d = 1 + 3 + 3 + 9 = 16$.

Thus, the correct answer is **D**.

21. The first 2007 positive integers are each written in base $3.$ How many of these base-3 representations are palindromes? (A palindrome is a number that reads the same forward and backward.)

A 100

B 101

C 102

D 103

E 104

Solution:

A palindrome is fixed by its first half. Counting base-3 palindromes by length gives 2 of length 1 or $2, 6$ of length 3 or $4, 18$ of length 5 or $6,$ and 54 of length $7.$

That totals $2+2+6+6+18+18+54=106$ palindromes with at most 7 digits. Since $2007=2202100_3,$ the 7-digit palindromes larger than it are $2210122,$ $2211122, 2212122, 2220222, 2221222,$ and $2222222,$ which is 6 of them.

Therefore the count is $106-6=100.$

Thus, the correct answer is **A**.

22. Two particles move along the edges of equilateral $\triangle ABC$ in the direction

$$A \to B \to C \to A,$$

starting simultaneously and moving at the same speed. One starts at A, and the other starts at the midpoint of $\overline{BC}$. The midpoint of the line segment joining the two particles traces out a path that encloses a region R. What is the ratio of the area of R to the area of $\triangle ABC$?

A $\frac{1}{16}$

B $\frac{1}{12}$

C $\frac{1}{9}$

D $\frac{1}{6}$

E $\frac{1}{4}$

Solution:

Track a third point always halfway between the two particles. Between the moments when the particles are at vertices/midpoints, both particles move linearly, so the midpoint moves linearly too, tracing straight segments that form a small triangle XYZ.

By symmetry XYZ shares its center with $\triangle ABC$. If O is that center and F is the midpoint of a side, then

$$OZ = OC - ZC = \frac{2}{3}CF - \frac{1}{2}CF = \frac{1}{6}CF,$$

while $OC = \frac{2}{3}CF$.

So the ratio of circumradii is $\frac{OZ}{OC} = \frac{1}{4}$, and the area ratio is $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$.

Thus, the correct answer is **A**.

23. How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?

A 6

B 7

C 8

D 10

E 12

Solution:

Let the legs be $a \leq b$. The condition is $\frac{1}{2}ab = 3\left(a + b + \sqrt{a^2 + b^2}\right)$, so

$$ab - 6a - 6b = 6\sqrt{a^2 + b^2}.$$

Squaring and simplifying gives $ab(ab - 12a - 12b + 72) = 0$, hence $(a - 12)(b - 12) = 72$. The positive integer solutions are $(a, b) =$
$(3, 4), (13, 84), (14, 48), (15, 36), (16, 30), (18, 24), (20, 21).$

The pair $(3, 4)$ is extraneous (its area 6 does not equal 3 times its perimeter 12), so exactly 6 triangles work.

Thus, the correct answer is **A**.

24. How many pairs of positive integers (a, b) are there such that $\gcd(a, b) = 1$ and

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer?

A 4

B 6

C 9

D 12

E infinitely many

Solution:

Multiplying by b and subtracting a gives $\frac{14b^2}{9a} = bk - a$, an integer. Since $\gcd(a, b) = 1$, it follows that $a \mid 14$. Multiplying instead by $9a$ and subtracting $14b$ gives $\frac{9a^2}{b} = 9ak - 14b$, so $b \mid 9$.

Thus $a \in \{1, 2, 7, 14\}$ and $b \in \{1, 3, 9\}$. Checking the coprime candidates, the expression is an integer only for $(a, b) = (1, 3), (2, 3), (7, 3), (14, 3)$.

So there are 4 such pairs.

Thus, the correct answer is **A**.

25. Points $A, B, C, D,$ and E are located in 3-dimensional space with $AB = BC = CD = DE = EA = 2$ and $\angle ABC = \angle CDE = \angle DEA = 90^\circ$. The plane of $\triangle ABC$ is parallel to $\overline{DE}$. What is the area of $\triangle BDE$?

A $\sqrt{2}$

B $\sqrt{3}$

C 2

D $\sqrt{5}$

E $\sqrt{6}$

Solution:

Set $D = (-1, 0, 0)$ and $E = (1, 0, 0)$, and let $\triangle ABC$ lie in the plane $z = k > 0$. Because $\angle CDE$ and $\angle DEA$ are right angles, A and C lie on radius-2 circles centered at E and D in the planes $x = 1$ and $x = -1$, so $A = (1, y_1, k),\ C = (-1, y_2, k)$ with $y_j = \pm\sqrt{4 - k^2}$.

Since $\angle ABC = 90^\circ$, $AC = 2\sqrt{2}$, which forces $y_1 = -y_2$. Taking $y_1 = 1, y_2 = -1$, gives $k = \sqrt{3}$, so $A = (1, 1, \sqrt{3}), C = (-1, -1, \sqrt{3})$, and B is $(1, -1, \sqrt{3})$ or $(-1, 1, \sqrt{3})$.

In the first case $BE = 2$ with $BE \perp DE$; in the second $BD = 2$ with $BD \perp DE$. Either way $\triangle BDE$ has legs 2 and 2, so its area is $\frac{1}{2}(2)(2) = 2$.

Thus, the correct answer is **C**.

Problems: https://live.poshenloh.com/past-contests/amc12/2007B