2007 AMC 12A Solutions

Typeset by: LIVE by Po-Shen Loh
https://live.poshenloh.com/past-contests/amc12/2007A/solutions

1. One ticket to a show costs $\$20$ at full price. Susan buys 4 tickets using a coupon that gives her a 25% discount. Pam buys 5 tickets using a coupon that gives her a 30% discount. How many more dollars does Pam pay than Susan?

- A 2
- B 5
- C 10
- D 15
- E 20

Solution:

Susan pays $(4)(0.75)(20) = 60$ dollars.

Pam pays $(5)(0.70)(20) = 70$ dollars.

So Pam pays $70 - 60 = 10$ more dollars than Susan.

Thus, the correct answer is **C**.

2. An aquarium has a rectangular base that measures 100 cm by 40 cm and has a height of 50 cm. It is filled with water to a height of 40 cm. A brick with a rectangular base that measures 40 cm by 20 cm and a height of 10 cm is placed in the aquarium. By how many centimeters does the water rise?

A 0.5

B 1

C 1.5

D 2

E 2.5

Solution:

The brick has a volume of $40 \cdot 20 \cdot 10 = 8000$ cubic centimeters.

If the water rises by h centimeters, the added volume is $100 \cdot 40 \cdot h = 4000h$ cubic centimeters.

Setting this equal to the brick's volume gives $8000 = 4000h$, so $h = 2$.

Thus, the correct answer is **D**.

3. The larger of two consecutive odd integers is three times the smaller. What is their sum?

A 4

B 8

C 12

D 16

E 20

Solution:

Let the smaller integer be x. Then the larger is $x + 2$.

So $x + 2 = 3x$, which gives $x = 1$.

The two integers are 1 and 3, and their sum is 4.

Thus, the correct answer is **A**.

4. Kate rode her bicycle for 30 minutes at a speed of 16 mph, then walked for 90 minutes at a speed of 4 mph. What was her overall average speed in miles per hour?

- A 7
- B 9
- C 10
- D 12
- E 14

Solution:

Kate rode for $\frac{1}{2}$ hour at 16 mph, covering 8 miles.

She walked for $\frac{3}{2}$ hours at 4 mph, covering 6 miles.

She covered 14 miles in 2 hours, so her average speed was 7 mph.

Thus, the correct answer is **A**.

5. Last year Mr. John Q. Public received an inheritance. He paid 20% in federal taxes on the inheritance, and paid 10% of what he had left in state taxes. He paid a total of $\$10{,}500$ for both taxes. How many dollars was the inheritance?

- A $30{,}000$
- B $32{,}500$
- C $35{,}000$
- D $37{,}500$
- E $40{,}000$

Solution:

After federal taxes, Mr. Public keeps 80% of his inheritance.

He pays 10% of that in state taxes, which is 8% of the inheritance.

His total tax is $20\% + 8\% = 28\%$ of the inheritance, so the inheritance is $\$10{,}500/0.28 = \$37{,}500.$

Thus, the correct answer is **D**.

6. Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside $\triangle ABC$, $\angle ABC = 40^\circ$, and $\angle ADC = 140^\circ$. What is the degree measure of $\angle BAD$?

A 20

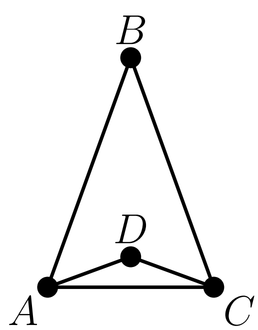
B 30

C 40

D 50

E 60

Solution:



Since $\triangle ABC$ is isosceles, $\angle BAC = \frac{1}{2}(180^\circ - \angle ABC) = 70^\circ$.

Since $\triangle ADC$ is isosceles, $\angle DAC = \frac{1}{2}(180^\circ - \angle ADC) = 20^\circ$.

Therefore $\angle BAD = \angle BAC - \angle DAC = 70^\circ - 20^\circ = 50^\circ$.

Thus, the correct answer is **D**.

7. Let $a, b, c, d,$ and e be five consecutive terms in an arithmetic sequence, and suppose that $a+b+c+d+e=30$. Which of the following can be found?

A a

B b

C c

D d

E e

Solution:

Let D be the common difference. Then $a=c-2D, b=c-D, d=c+D,$ and $e=c+2D,$ so

$$a+b+c+d+e=5c.$$

Thus $5c=30$, giving $c=6$.

The other terms cannot be determined: the sequences $4,5,6,7,8$ and $10,8,6,4,2$ both satisfy the conditions but differ in every term except the middle one.

Thus, the correct answer is **C**.

8. A star-polygon is drawn on a clock face by drawing a chord from each number to the fifth number counted clockwise from that number. That is, chords are drawn from 12 to $5,$ from 5 to $10,$ from 10 to $3,$ and so on, ending back at $12.$ What is the degree measure of the angle at each vertex in the star-polygon?

- A 20
- B 24
- C 30
- D 36
- E 60

Solution:

Consider the two chords meeting at the number $5.$ They run to 12 and to $10,$ so the arc they subtend extends from 10 to $12.$

That arc spans two of the twelve hour-marks, so its measure is $\frac{2}{12} \cdot 360^\circ = 60^\circ.$

By the Inscribed Angle Theorem, the vertex angle is half the arc, or $\frac{1}{2} \cdot 60^\circ = 30^\circ.$ By symmetry every vertex angle equals $30^\circ.$

Thus, the correct answer is **C**.

9. Yan is somewhere between his home and the stadium. To get to the stadium he can walk directly to the stadium, or else he can walk home and then ride his bicycle to the stadium. He rides 7 times as fast as he walks, and both choices require the same amount of time. What is the ratio of Yan's distance from his home to his distance from the stadium?

A $\frac{2}{3}$

B $\frac{3}{4}$

C $\frac{4}{5}$

D $\frac{5}{6}$

E $\frac{6}{7}$

Solution:

Let w be the walking speed and let x and y be Yan's distances from home and from the stadium.

Walking to the stadium takes $\frac{y}{w}$. Walking home then biking takes $\frac{x}{w}+\frac{x+y}{7w}=\frac{8x+y}{7w}$.

Setting these equal gives $7y=8x+y$, so $8x=6y$ and $\frac{x}{y}=\frac{3}{4}$.

Thus, the correct answer is **B**.

10. A triangle with side lengths in the ratio $3:4:5$ is inscribed in a circle of radius 3. What is the area of the triangle?

A 8.64

B 12

C 5π

D 17.28

E 18

Solution:

Let the sides be $3x, 4x$, and $5x$. The triangle is right, so its hypotenuse is a diameter.

Thus $5x = 2 \cdot 3 = 6$, giving $x = \frac{6}{5}$.

The area is $\frac{1}{2} \cdot 3x \cdot 4x = 6x^2 = 6 \cdot \frac{36}{25} = \frac{216}{25} = 8.64$.

Thus, the correct answer is **A**.

11. A finite sequence of three-digit integers has the property that the tens and units digits of each term are, respectively, the hundreds and tens digits of the next term, and the tens and units digits of the last term are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with terms $247, 475,$ and 756 and end with the term $824.$ Let S be the sum of all the terms in the sequence. What is the largest prime number that always divides S?

A	3
B	7
C	13
D	37
E	43

Solution:

Because of the cycling property, each digit that appears is used the same number of times in the hundreds, tens, and units places.

Let k be the sum of the units digits over all terms. Then $S = 100k + 10k + k = 111k = 3 \cdot 37 \cdot k.$

So S is always divisible by $37.$ It need not be divisible by anything larger: the sequence $123, 231, 312$ gives $S = 666 = 2 \cdot 3^2 \cdot 37.$

Thus, the correct answer is **D**.

12. Integers $a, b, c,$ and $d,$ not necessarily distinct, are chosen independently and at random from 0 to $2007,$ inclusive. What is the probability that $ad - bc$ is even?

A $\frac{3}{8}$

B $\frac{7}{16}$

C $\frac{1}{2}$

D $\frac{9}{16}$

E $\frac{5}{8}$

Solution:

Exactly half of the integers from 0 to 2007 are odd.

A product ad is odd only when both factors are odd, with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, and even with probability $\frac{3}{4}$. The same holds for bc.

Then $ad - bc$ is even when both products are odd or both are even:

$$\frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} = \frac{10}{16} = \frac{5}{8}.$$

Thus, the correct answer is **E**.

13. A piece of cheese is located at $(12, 10)$ in a coordinate plane. A mouse is at $(4, -2)$ and is running up the line $y = -5x + 18$. At the point (a, b) the mouse starts getting farther from the cheese rather than closer to it. What is $a + b$?

- A 6
- B 10
- C 14
- D 18
- E 22

Solution:

The mouse is closest to the cheese at the foot of the perpendicular from $(12, 10)$ to the line.

This perpendicular has slope $\frac{1}{5}$, so its equation is $y = 10 + \frac{1}{5}(x - 12) = \frac{1}{5}x + \frac{38}{5}$. Setting $\frac{1}{5}x + \frac{38}{5} = -5x + 18$ gives $x = 2$ and $y = 8$. Thus $(a, b) = (2, 8)$ and $a + b = 10$.

Thus, the correct answer is **B**.

14. Let $a, b, c, d,$ and e be distinct integers such that

$$(6-a)(6-b)(6-c)(6-d)(6-e) = 45.$$

What is $a+b+c+d+e$?

A 5

B 17

C 25

D 27

E 30

Solution:

The five factors are distinct integers multiplying to 45. If any factor had absolute value more than 5, the remaining four (distinct) would have product at least $|(-3)(-1)(1)(3)| = 9$, forcing the total above 45.

So the factors come from $\pm 1, \pm 3, \pm 5$. The product of all six of these is $-225 = (-5)(45)$, so the five factors are $-3, -1, 1, 3, 5$.

Then a, b, c, d, e are $9, 7, 5, 3, 1$ in some order, and their sum is 25.

Thus, the correct answer is **C**.

15. The set $\{3, 6, 9, 10\}$ is augmented by a fifth element n, not equal to any of the other four. The median of the resulting set is equal to its mean. What is the sum of all possible values of n?

A 7

B 9

C 19

D 24

E 26

Solution:

The mean is $\frac{28+n}{5}$.

If $n < 6$, the median is 6, so $28 + n = 30$ and $n = 2$.

If $6 < n < 9$, the median is n, so $28 + n = 5n$ and $n = 7$.

If $n > 9$, the median is 9, so $28 + n = 45$ and $n = 17$.

The sum of all possible values is $2 + 7 + 17 = 26$.

Thus, the correct answer is **E**.

16. How many three-digit numbers are composed of three distinct digits such that one digit is the average of the other two?

- A 96
- B 104
- C 112
- D 120
- E 256

Solution:

The three distinct digits form an increasing arithmetic progression. Counting by common difference: 8 with difference $1, 6$ with difference $2, 4$ with difference $3,$ and 2 with difference $4,$ for 20 sets.

Of these, 4 sets contain 0 (namely $\{0,1,2\},\{0,2,4\},\{0,3,6\},\{0,4,8\}$); each yields $2\cdot 2! = 4$ valid numbers since 0 cannot lead.

The other 16 sets each yield $3! = 6$ numbers. The total is $4\cdot 4 + 16\cdot 6 = 112.$

Thus, the correct answer is **C**.

17. Suppose that $\sin a + \sin b = \sqrt{\frac{5}{3}}$ and $\cos a + \cos b = 1$. What is $\cos(a-b)$?

A $\sqrt{\dfrac{5}{3}} - 1$

B $\dfrac{1}{3}$

C $\dfrac{1}{2}$

D $\dfrac{2}{3}$

E 1

Solution:

Squaring both equations gives $\sin^2 a + 2\sin a \sin b + \sin^2 b = \frac{5}{3}$ and $\cos^2 a + 2\cos a \cos b + \cos^2 b = 1$.

Adding and using $\sin^2\theta + \cos^2\theta = 1$ twice,

$$2 + 2(\sin a \sin b + \cos a \cos b) = \tfrac{8}{3}.$$

So $\cos(a-b) = \sin a \sin b + \cos a \cos b = \frac{1}{3}$.

Thus, the correct answer is **B**.

18. The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has real coefficients, and $f(2i) = f(2+i) = 0$. What is $a+b+c+d$?

A 0

B 1

C 4

D 9

E 16

Solution:

Since f has real coefficients, the conjugates $-2i$ and $2-i$ are also roots. Thus

$$f(x) = (x^2+4)(x^2-4x+5) = x^4 - 4x^3 + 9x^2 - 16x + 20.$$

Then $a+b+c+d = -4+9-16+20 = 9$. Equivalently, $a+b+c+d = f(1) - 1 = (1+4)(1+1) - 1 = 9$.

Thus, the correct answer is **D**.

19. Triangles ABC and ADE have areas 2007 and $7002,$ respectively, with $B = (0,0),$ $C = (223,0), D = (680,380),$ and $E = (689,389).$ What is the sum of all possible x -coordinates of A?

A 282

B 300

C 600

D 900

E 1200

Solution:

The altitude h from A in $\triangle ABC$ satisfies $2007 = \frac{1}{2} \cdot 223 \cdot h,$ so $h = 18.$ Thus A lies on $y = 18$ or $y = -18.$

Line DE has equation $x - y - 300 = 0.$ The condition on $\triangle ADE$ similarly places A on one of two lines parallel to $DE.$

The four possible positions of A are the vertices of a parallelogram whose center is the intersection of $y = 0$ with line $DE,$ namely $(300,0).$ Hence the sum of the four x-coordinates is $4 \cdot 300 = 1200.$

Thus, the correct answer is **E**.

20. Corners are sliced off a unit cube so that the six faces each become regular octagons. What is the total volume of the removed tetrahedra?

A $\frac{5\sqrt{2}-7}{3}$

B $\frac{10-7\sqrt{2}}{3}$

C $\frac{3-2\sqrt{2}}{3}$

D $\frac{8\sqrt{2}-11}{3}$

E $\frac{6-4\sqrt{2}}{3}$

Solution:

Slicing removes two equal segments of length x from each edge. Each octagon then has side length $x\sqrt{2}$, and the edge satisfies $1 = 2x + x\sqrt{2}$, so

$$x = \frac{1}{2+\sqrt{2}} = \frac{2-\sqrt{2}}{2}.$$

Each removed corner is a tetrahedron with three mutually perpendicular legs of length x, so its volume is $\frac{1}{6}x^3$. There are 8 corners, giving total volume

$$8 \cdot \tfrac{1}{6}x^3 = \tfrac{4}{3}\left(\tfrac{2-\sqrt{2}}{2}\right)^3 = \frac{10-7\sqrt{2}}{3}.$$

Thus, the correct answer is **B**.

21. The sum of the zeros, the product of the zeros, and the sum of the coefficients of the function $f(x) = ax^2 + bx + c$ are equal. Their common value must also be which of the following?

A the coefficient of x^2

B the coefficient of x

C the y-intercept of the graph of $y = f(x)$

D one of the x-intercepts of the graph of $y = f(x)$

E the mean of the x-intercepts of the graph of $y = f(x)$

Solution:

The product of the zeros is $\frac{c}{a}$ and the sum of the zeros is $-\frac{b}{a}$. Equating them gives $c = -b$.

Then the sum of the coefficients is $a + b + c = a$, which is the coefficient of x^2.

The other choices fail in general: for $f(x) = -2x^2 - 4x + 4$ the common value is -2, but the coefficient of x is -4, the y-intercept is 4, the x-intercepts are $-1 \pm \sqrt{3}$, and their mean is -1.

Thus, the correct answer is **A**.

22. For each positive integer n, let $S(n)$ denote the sum of the digits of n. For how many values of n is $n + S(n) + S(S(n)) = 2007$?

- A 1
- B 2
- C 3
- D 4
- E 5

Solution:

For $n \leq 2007, S(n) \leq S(1999) = 28$, and then $S(S(n)) \leq S(28) = 10$. So any solution has $n \geq 2007 - 28 - 10 = 1969$.

Also $n, S(n)$, and $S(S(n))$ are congruent modulo 9, and 2007 is a multiple of 9, so all three must be multiples of 3.

Checking the multiples of 3 between 1969 and 2007 (many are eliminated because $n + S(n)$ already exceeds 2007) leaves $1977, 1980, 1983$, and 2001. That is 4 values.

Thus, the correct answer is **D**.

23. Square $ABCD$ has area 36, and AB is parallel to the x-axis. Vertices $A, B,$ and C are on the graphs of $y = \log_a x, y = 2\log_a x,$ and $y = 3\log_a x,$ respectively. What is a?

A $\sqrt[6]{3}$

B $\sqrt{3}$

C $\sqrt[3]{6}$

D $\sqrt{6}$

E 6

Solution:

Let $A = (p, \log_a p)$ and $B = (q, 2\log_a q)$. Since AB is horizontal, $\log_a p = 2\log_a q = \log_a q^2$, so $p = q^2$.

The side length is $6 = |p - q| = |q^2 - q|$, whose only positive solution is $q = 3$.

Since $C = (q, 3\log_a q)$, the vertical side gives $BC = 6 = \log_a q = \log_a 3$. Thus $a^6 = 3$, so $a = \sqrt[6]{3}$.

Thus, the correct answer is **A**.

24. For each integer $n > 1$, let $F(n)$ be the number of solutions of the equation $\sin x = \sin nx$ on the interval $[0, \pi]$. What is $\sum_{n=2}^{2007} F(n)$?

A $2{,}014{,}524$

B $2{,}015{,}028$

C $2{,}015{,}033$

D $2{,}016{,}532$

E $2{,}017{,}033$

Solution:

On each interval where $\sin nx \geq 0$, the graphs of $\sin x$ and $\sin nx$ meet twice, unless they share the value 1 there, in which case they meet once. Counting the humps and the endpoint at $(\pi, 0)$ gives

$F(n) = n + 1$ when n is even or $n \equiv 3 \pmod 4$, and $F(n) = n$ when $n \equiv 1 \pmod 4$.

Thus

$$\sum_{n=2}^{2007} F(n) = \sum_{n=2}^{2007} (n+1) - \#\{n \equiv 1 \pmod 4\}.$$

The first sum is $2{,}017{,}033$, and there are 501 values $n \equiv 1 \pmod 4$ in the range, giving $2{,}017{,}033 - 501 = 2{,}016{,}532$.

Thus, the correct answer is **D**.

25. Call a set of integers *spacy* if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacy?

- A 121
- B 123
- C 125
- D 127
- E 129

Solution:

Let c_n be the number of spacy subsets of $\{1, \ldots, n\}$. A spacy subset either omits n (there are c_{n-1} of these) or contains n, in which case it omits $n-1$ and $n-2$ (there are c_{n-3} of these).

Hence $c_n = c_{n-1} + c_{n-3}$, with $c_1 = 2, c_2 = 3, c_3 = 4.$

The sequence continues $6, 9, 13, 19, 28, 41, 60, 88, 129$, so $c_{12} = 129.$

Thus, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/2007A