# 2006 AMC 12B Solutions

Typeset by: LIVE by Po-Shen Loh
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**1.** What is

$$(-1)^1 + (-1)^2 + \cdots + (-1)^{2006}?$$

A $-2006$

B $-1$

C $0$

D $1$

E $2006$

## Solution:

Since $(-1)^k = -1$ for odd $k$ and $(-1)^k = 1$ for even $k$, the terms alternate $-1, 1, -1, 1, \ldots$

There are $2006$ terms, forming $1003$ pairs, each equal to $(-1) + 1 = 0$. The total is $0$.

Thus, the correct answer is **C**.

**2.** For real numbers $x$ and $y$, define

$$x\spadesuit y = (x+y)(x-y).$$

What is $3\spadesuit(4\spadesuit 5)$?

A $-72$

B $-27$

C $-24$

D $24$

E $72$

## Solution:

Since $x\spadesuit y = x^2 - y^2$, the inner value is $4\spadesuit 5 = 16 - 25 = -9.$

Then $3\spadesuit(-9) = 3^2 - (-9)^2 = 9 - 81 = -72.$

Thus, the correct answer is **A**.

**3.** A football game was played between two teams, the Cougars and the Panthers. The two teams scored a total of $34$ points, and the Cougars won by a margin of $14$ points. How many points did the Panthers score?

- A $10$
- B $14$
- C $17$
- D $20$
- E $24$

## Solution:

Let $c$ and $p$ be the Cougars' and Panthers' scores. Then $c + p = 34$ and $c - p = 14.$

Subtracting gives $2p = 20,$ so $p = 10.$

Thus, the correct answer is **A**.

4. Mary is about to pay for five items at the grocery store. The prices of the items are $\$7.99, \$4.99, \$2.99, \$1.99,$ and $\$0.99.$ Mary will pay with a twenty-dollar bill. Which of the following is closest to the percentage of the $\$20.00$ that she will receive in change?

A $5$

B $10$

C $15$

D $20$

E $25$

## Solution:

The five prices total about $8+5+3+2+1=19$ dollars, so the change is about $\$1.00.$

This is

$$\frac{1}{20}=5\%$$

of the twenty-dollar bill.

Thus, the correct answer is **A**.

**5.** John is walking east at a speed of $3$ miles per hour, while Bob is also walking east, but at a speed of $5$ miles per hour. If Bob is now $1$ mile west of John, how many minutes will it take for Bob to catch up to John?

- A $30$
- B $50$
- C $60$
- D $90$
- E $120$

## Solution:

Bob closes the gap at a relative speed of $5 - 3 = 2$ miles per hour. To cover the $1$-mile gap takes

$$\frac{1}{2}\ \text{hour} = 30\ \text{minutes}.$$

Thus, the correct answer is **A**.

**6.** Francesca uses $100$ grams of lemon juice, $100$ grams of sugar, and $400$ grams of water to make lemonade. There are $25$ calories in $100$ grams of lemon juice and $386$ calories in $100$ grams of sugar. Water contains no calories. How many calories are in $200$ grams of her lemonade?

- A $129$
- **B** $137$
- C $174$
- D $223$
- E $411$

## Solution:

The full batch weighs $100 + 100 + 400 = 600$ grams and contains $25 + 386 = 411$ calories.

Since $200$ grams is one third of the batch, it has

$$\frac{411}{3} = 137$$

calories.

Thus, the correct answer is **B**.

**7.** Mr. and Mrs. Lopez have two children. When they get into their family car, two people sit in the front, and the other two sit in the back. Either Mr. Lopez or Mrs. Lopez must sit in the driver's seat. How many seating arrangements are possible?

- A $4$
- B $12$
- C $16$
- D $24$
- E $48$

## Solution:

The driver is one of the two parents: $2$ choices.

Any of the remaining $3$ people can sit in the front passenger seat, and the last $2$ people fill the back in $2$ orders.

The total is $2 \cdot 3 \cdot 2 = 12.$

Thus, the correct answer is **B**.

**8.** The lines

$$x = \tfrac{1}{4}y + a \quad \text{and} \quad y = \tfrac{1}{4}x + b$$

intersect at the point $(1, 2)$. What is $a + b$?

A $0$

B $\frac{3}{4}$

C $1$

D $2$

E $\frac{9}{4}$

## Solution:

Substituting $(1, 2)$ gives

$$1 = \frac{2}{4} + a \quad \Rightarrow \quad a = \frac{1}{2},$$

and

$$2 = \frac{1}{4} + b \quad \Rightarrow \quad b = \frac{7}{4}.$$

Therefore

$$a + b = \frac{1}{2} + \frac{7}{4} = \frac{9}{4}.$$

Thus, the correct answer is **E**.

**9.** How many even three-digit integers have the property that their digits, read left to right, are in strictly increasing order?

- A $21$
- B $34$
- C $51$
- D $72$
- E $150$

## Solution:

Let the digits be $a < b < c$ with $c$ even. Since $a \geq 1$, no digit is zero, and $c \neq 2$ (there is no room for two smaller nonzero digits).

Once the units digit $c$ is fixed, any two distinct digits below it can be arranged in increasing order in exactly one way. So the count for each $c$ is $\binom{c-1}{2}$.

For $c = 4, 6, 8$ this gives

$$\binom{3}{2} + \binom{5}{2} + \binom{7}{2} = 3 + 10 + 21 = 34.$$

Thus, the correct answer is **B**.

**10.** In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is $15$. What is the greatest possible perimeter of the triangle?

- A $43$
- B $44$
- C $45$
- D $46$
- E $47$

## Solution:

Let the sides be $x, 3x,$ and $15$. The triangle inequality requires $x + 3x > 15$, so $x \geq 4$, and $x + 15 > 3x$, so $x \leq 7$.

The perimeter $4x + 15$ is largest when $x = 7$, giving $7 + 21 + 15 = 43$.

Thus, the correct answer is **A**.

**11.** Joe and JoAnn each bought $12$ ounces of coffee in a $16$-ounce cup. Joe drank $2$ ounces of his coffee and then added $2$ ounces of cream. JoAnn added $2$ ounces of cream, stirred the coffee well, and then drank $2$ ounces. What is the resulting ratio of the amount of cream in Joe's coffee to that in JoAnn's coffee?

A $\frac{6}{7}$

B $\frac{13}{14}$

C $1$

D $\frac{14}{13}$

E $\frac{7}{6}$

## Solution:

Joe adds the cream last, so his cup holds all $2$ ounces of cream.

JoAnn's cup has $14$ ounces of mixture containing $2$ ounces of cream. Drinking $2$ ounces removes a fraction $\frac{2}{14}$ of everything, leaving

$$2 \cdot \frac{12}{14} = \frac{12}{7}$$

ounces of cream.

The ratio is

$$\frac{2}{12/7} = \frac{14}{12} = \frac{7}{6}.$$

Thus, the correct answer is **E**.

**12.** The parabola $y = ax^2 + bx + c$ has vertex $(p, p)$ and $y$-intercept $(0, -p)$, where $p \neq 0$. What is $b$?

A $-p$

B $0$

C $2$

D $4$

E $p$

## Solution:

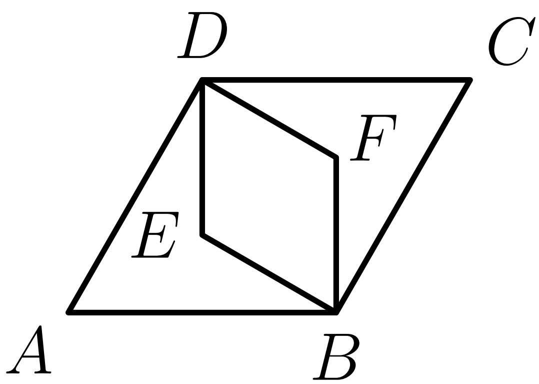
The vertex form is $y = a(x - p)^2 + p$.

At $x = 0, y = ap^2 + p = -p$, so $ap^2 = -2p$ and $a = -\frac{2}{p}$.

Expanding, $y = ax^2 - 2ap\,x + ap^2 + p$, so $b = -2ap = -2\left(-\frac{2}{p}\right)p = 4.$

Thus, the correct answer is **D**.

**13.** Rhombus $ABCD$ is similar to rhombus $BFDE$. The area of rhombus $ABCD$ is $24$, and $\angle BAD = 60°$. What is the area of rhombus $BFDE$?




A $6$

B $4\sqrt{3}$

C $8$

D $9$

E $6\sqrt{3}$

## Solution:

Because $\angle BAD = 60°$ and $AB = AD$, triangle $ABD$ is equilateral. The diagonals $AC$ and $BD$ together with segments $BE, DF$ split $ABCD$ into six congruent triangles.

Each of these triangles has area $\frac{24}{6} = 4$.

Rhombus $BFDE$ is the union of $\triangle BED$ and $\triangle BFD$, two of them, so its area is $8$.

Thus, the correct answer is **C**.

**14.** Elmo makes $N$ sandwiches for a fundraiser. For each sandwich he uses $B$ globs of peanut butter at $4¢$ per glob and $J$ blobs of jam at $5¢$ per blob. The cost of the peanut butter and jam to make all the sandwiches is $\$2.53$. Assume that $B$, $J$, and $N$ are positive integers with $N > 1$. What is the cost of the jam Elmo uses to make the sandwiches?

A $\$1.05$

B $\$1.25$

C $\$1.45$

D $\$1.65$

E $\$1.85$

## Solution:

The total cost in cents is $N(4B+5J) = 253 = 11 \cdot 23$. Since $N > 1$, the value of $N$ is $11, 23$, or $253$.

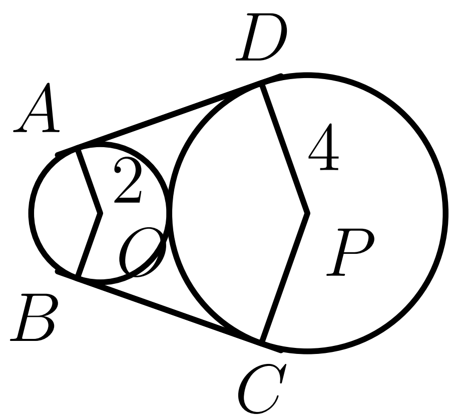
If $N = 253$ then $4B + 5J = 1$, and if $N = 23$ then $4B + 5J = 11$; neither has a positive integer solution.

So $N = 11$ and $4B + 5J = 23$, whose only positive solution is $B = 2, J = 3$.

The jam costs $N \cdot J \cdot 5¢ = 11 \cdot 3 \cdot 5 = 165$ cents, or $\$1.65$.

Thus, the correct answer is **D**.

**15.** Circles with centers $O$ and $P$ have radii $2$ and $4$, respectively, and are externally tangent. Points $A$ and $B$ are on the circle centered at $O$, and points $C$ and $D$ are on the circle centered at $P$, such that $AD$ and $BC$ are common external tangents to the circles. What is the area of hexagon $AOBCPD$?




A $18\sqrt{3}$

B $24\sqrt{2}$

C $36$

D $24\sqrt{3}$

E $32\sqrt{2}$

## Solution:

The circles are externally tangent, so $OP = 2 + 4 = 6$. In quadrilateral $AOPD$, both $OA = 2$ and $PD = 4$ are perpendicular to the tangent line $AD$, making it a right trapezoid.

Drawing the line through $O$ parallel to $AD$ creates a right triangle with hypotenuse $OP = 6$ and one leg $PD - OA = 2$, so $AD = \sqrt{6^2 - 2^2} = \sqrt{32} = 4\sqrt{2}$.

The trapezoid $AOPD$ has area

$$\frac{1}{2}(2 + 4)(4\sqrt{2}) = 12\sqrt{2}.$$

By symmetry the hexagon $AOBCPD$ is made of two such trapezoids, so its area is $2 \cdot 12\sqrt{2} = 24\sqrt{2}$.

Thus, the correct answer is **B**.

**16.** Regular hexagon $ABCDEF$ has vertices $A$ and $C$ at $(0, 0)$ and $(7, 1)$, respectively. What is its area?

A $20\sqrt{3}$

B $22\sqrt{3}$

C $25\sqrt{3}$

D $27\sqrt{3}$

E $50$

## Solution:

The distance is $AC = \sqrt{7^2 + 1^2} = \sqrt{50}$. In a regular hexagon with side $s$, the distance between vertices two apart is $s\sqrt{3}$, so $s^2 \cdot 3 = 50$, giving $s^2 = \frac{50}{3}$.

The hexagon's area is

$$\frac{3\sqrt{3}}{2}s^2 = \frac{3\sqrt{3}}{2} \cdot \frac{50}{3} = 25\sqrt{3}.$$

Thus, the correct answer is **C**.

**17.** For a particular peculiar pair of dice, the probabilities of rolling $1, 2, 3, 4, 5,$ and $6$ on each die are in the ratio $1 : 2 : 3 : 4 : 5 : 6.$ What is the probability of rolling a total of $7$ on the two dice?

A $\dfrac{4}{63}$

B $\dfrac{1}{8}$

C $\dfrac{8}{63}$

D $\dfrac{1}{6}$

E $\dfrac{2}{7}$

## Solution:

Since the weights sum to $21,$ the probability of rolling $k$ is $\dfrac{k}{21}.$

A total of $7$ comes from $(1, 6), (2, 5), \ldots, (6, 1),$ so the probability is

$$\frac{1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4 + 4 \cdot 3 + 5 \cdot 2 + 6 \cdot 1}{21^2} = \frac{56}{441} = \frac{8}{63}.$$

Thus, the correct answer is **C**.

**18.** An object in the plane moves from one lattice point to another. At each step, the object may move one unit to the right, one unit to the left, one unit up, or one unit down. If the object starts at the origin and takes a ten-step path, how many different points could be the final point?

A $120$

B $121$

C $221$

D $230$

E $231$

## Solution:

Each step changes the coordinate sum by $1$, so after $10$ steps the endpoint $(a, b)$ has $a + b$ even, and $|a| + |b| \leq 10$. Any such point is reachable: walk $|a| + |b|$ steps to it, then use the remaining even number of steps going out and back.

The reachable points lie on the lines $a + b = 2k$ for $-5 \leq k \leq 5$. Each such line meets the diamond in exactly $11$ lattice points.

With $11$ lines and $11$ points each, there are $121$ points.

Thus, the correct answer is **B**.

**19.** Mr. Jones has eight children of different ages. On a family trip his oldest child, who is $9$, spots a license plate with a $4$-digit number in which each of two digits appears two times. "Look, daddy!" she exclaims. "That number is evenly divisible by the age of each of us kids!" "That's right," replies Mr. Jones, "and the last two digits just happen to be my age." Which of the following is *not* the age of one of Mr. Jones's children?

- A $4$
- B $5$
- C $6$
- D $7$
- E $8$

## Solution:

The number has the form $aabb$, $abab$, or $baab$. Divisibility by $9$ means $2(a+b)$ is a multiple of $9$, so $a+b=9$.

The children include a $4$- or $8$-year-old, so the number is divisible by $4$. The possibilities become $1188, 2772, 3636, 5544, 6336, 7272, 9900$.

Since the last two digits are Mr. Jones's age, $9900$ is impossible, and none of the others is a multiple of $5$. So the children's ages cannot include $5$. Indeed $5544$ is divisible by $1, 2, 3, 4, 6, 7, 8, 9$.

Thus, the correct answer is **B**.

**20.** Let $x$ be chosen at random from the interval $(0, 1)$. What is the probability that

$$\lfloor \log_{10} 4x \rfloor - \lfloor \log_{10} x \rfloor = 0?$$

Here $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to $x$.

A $\frac{1}{8}$

B $\frac{3}{20}$

C $\frac{1}{6}$

D $\frac{1}{5}$

E $\frac{1}{4}$

## Solution:

The equation says $\lfloor \log_{10} x \rfloor = \lfloor \log_{10} 4x \rfloor$, i.e. $x$ and $4x$ lie in the same interval $[10^n, 10^{n+1})$.

This holds exactly when $10^n \leq x$ and $4x < 10^{n+1}$, that is $10^n \leq x < \dfrac{10^{n+1}}{4}$.

Within $[10^n, 10^{n+1})$, the favorable fraction is

$$\frac{10^{n+1}/4 - 10^n}{10^{n+1} - 10^n} = \frac{10/4 - 1}{10 - 1} = \frac{1}{6}.$$

Since this fraction is the same on every such interval, the overall probability is $\dfrac{1}{6}$.

Thus, the correct answer is **C**.

**21.** Rectangle $ABCD$ has area $2006$. An ellipse with area $2006\pi$ passes through $A$ and $C$ and has foci at $B$ and $D$. What is the perimeter of the rectangle? (The area of an ellipse is $\pi ab$, where $2a$ and $2b$ are the lengths of its axes.)

A $\frac{16\sqrt{2006}}{\pi}$

B $\frac{1003}{4}$

C $8\sqrt{1003}$

D $6\sqrt{2006}$

E $\frac{32\sqrt{1003}}{\pi}$

## Solution:

Let the rectangle's sides be $x$ and $y$. Point $A$ is on the ellipse with foci $B$ and $D$, so $x + y = AB + AD = 2a$. The distance between the foci is the diagonal, so $\sqrt{x^2 + y^2} = 2\sqrt{a^2 - b^2}$.

Then $2xy = (x + y)^2 - (x^2 + y^2) = 4a^2 - (4a^2 - 4b^2) = 4b^2$, so $xy = 2b^2$. The area gives $2b^2 = 2006$, hence $b^2 = 1003$.

The ellipse area gives $\pi ab = 2006\pi$, so $ab = 2006$ and $a = \frac{2006}{\sqrt{1003}} = 2\sqrt{1003}$.

The perimeter is $2(x + y) = 4a = 8\sqrt{1003}$.

Thus, the correct answer is **C**.

**22.** Suppose $a, b,$ and $c$ are positive integers with $a+b+c=2006,$ and $a!\,b!\,c!=m\cdot 10^n,$ where $m$ and $n$ are integers and $m$ is not divisible by $10.$ What is the smallest possible value of $n?$

- A $489$
- B $492$
- C $495$
- D $498$
- E $501$

## Solution:

Since factors of $2$ are more plentiful than factors of $5, n$ equals the number of factors of $5$ in $a!\,b!\,c!,$ namely

$$n=\sum_{k\geq 1}\left(\left\lfloor \frac{a}{5^k}\right\rfloor+\left\lfloor \frac{b}{5^k}\right\rfloor+\left\lfloor \frac{c}{5^k}\right\rfloor\right).$$

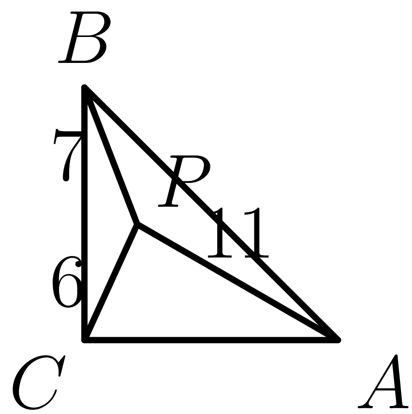
For each $k, \lfloor a/5^k\rfloor+\lfloor b/5^k\rfloor+\lfloor c/5^k\rfloor \geq \lfloor 2006/5^k\rfloor-2.$ Summing over $k=1,2,3,4$ (as $2006<5^5$) gives

$$n\geq(401+80+16+3)-4\cdot 2=492.$$

Equality is attainable, for example with $a=b=624$ and $c=758.$ So the minimum is $492.$

Thus, the correct answer is **B**.

**23.** Isosceles $\triangle ABC$ has a right angle at $C$. Point $P$ is inside $\triangle ABC$, such that $PA = 11$, $PB = 7$, and $PC = 6$. Legs $\overline{AC}$ and $\overline{BC}$ have length $s = \sqrt{a + b\sqrt{2}}$, where $a$ and $b$ are positive integers. What is $a + b$?




A $85$

B $91$

C $108$

D $121$

E $127$

## Solution:

Rotate $\triangle ABC$ by $90^\circ$ about $C$, sending $A$ to $B$ and $P$ to $P'$. Then $CP' = CP = 6$ and $\angle PCP' = 90^\circ$, so $\triangle PCP'$ is an isosceles right triangle with $PP' = 6\sqrt{2}$.

Also $BP' = AP = 11$. Since $(6\sqrt{2})^2 + 7^2 = 72 + 49 = 121 = 11^2$, triangle $BPP'$ has a right angle at $P$. Hence $\angle BPC = \angle BPP' + \angle P'PC = 90^\circ + 45^\circ = 135^\circ$.

By the Law of Cosines in $\triangle BPC$,

$$BC^2 = 6^2 + 7^2 - 2 \cdot 6 \cdot 7 \cos 135^\circ = 85 + 42\sqrt{2}.$$

So $s^2 = 85 + 42\sqrt{2}$, giving $a = 85$, $b = 42$, and $a + b = 127$.

Thus, the correct answer is **E**.

**24.** Let $S$ be the set of all points $(x, y)$ in the coordinate plane such that $0 \leq x \leq \dfrac{\pi}{2}$ and $0 \leq y \leq \dfrac{\pi}{2}$. What is the area of the subset of $S$ for which

$$\sin^2 x - \sin x \sin y + \sin^2 y \leq \frac{3}{4}?$$

A $\dfrac{\pi^2}{9}$

B $\dfrac{\pi^2}{8}$

C $\dfrac{\pi^2}{6}$

D $\dfrac{3\pi^2}{16}$

E $\dfrac{2\pi^2}{9}$

## Solution:

Fixing $y$, solve $\sin^2 x - \sin x \sin y + \sin^2 y = \dfrac{3}{4}$ as a quadratic in $\sin x$ :

$$\sin x = \frac{1}{2} \sin y \pm \frac{\sqrt{3}}{2} \cos y = \sin\left(y \pm \frac{\pi}{3}\right).$$

Within $S$, $\sin x = \sin\left(y - \frac{\pi}{3}\right)$ gives the line $x = y - \frac{\pi}{3}$, while $\sin x = \sin\left(y + \frac{\pi}{3}\right)$ gives $x = y + \frac{\pi}{3}$ for $y \leq \frac{\pi}{6}$ and $x = -y + \frac{2\pi}{3}$ for $y \geq \frac{\pi}{6}$.

These lines split $S$ into regions; testing the corners shows the inequality holds only in the middle band. Its area is

$$\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\left(\frac{\pi}{3}\right)^2 - 2 \cdot \frac{1}{2}\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{6}.$$

Thus, the correct answer is **C**.

**25.** A sequence $a_1, a_2, \ldots$ of non-negative integers is defined by the rule $a_{n+2} = |a_{n+1} - a_n|$ for $n \geq 1$. If $a_1 = 999$, $a_2 < 999$, and $a_{2006} = 1$, how many different values of $a_2$ are possible?

A $165$

B $324$

C $495$

D $499$

E $660$

## Solution:

The rule gives $a_n \equiv a_{n+3} \pmod 2$, so $a_2$ has the same parity as $a_{2006} = 1$; thus $a_2$ is odd.

Every term is a multiple of $\gcd(a_1, a_2)$, and $a_{2006} = 1$ forces $\gcd(999, a_2) = 1$. Since $999 = 3^3 \cdot 37$, we need $a_2$ not divisible by $3$ or $37$.

Among the odd integers in $[1, 998]$ there are $499$; removing the $166$ multiples of $3$ and $13$ multiples of $37$, then adding back the $4$ multiples of $111$, leaves

$$499 - 166 - 13 + 4 = 324.$$

Each such $a_2$ works: with $\gcd(a_1, a_2) = 1$ the sequence eventually cycles through $1, 1, 0$, and the parity condition makes $a_{2006} = 1$.

Thus, the correct answer is **B**.

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