2006 AMC 12A Solutions

Typeset by: LIVE by Po-Shen Loh
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1. Sandwiches at Joe's Fast Food cost $\$3$ each and sodas cost $\$2$ each. How many dollars will it cost to purchase 5 sandwiches and 8 sodas?

A 31

B 32

C 33

D 34

E 35

Solution:

Five sandwiches cost $5 \cdot 3 = 15$ dollars and eight sodas cost $8 \cdot 2 = 16$ dollars. Together they cost $15 + 16 = 31$ dollars.

Thus, the correct answer is **A**.

2. Define $x \otimes y = x^3 - y$. What is $h \otimes (h \otimes h)$?

A $-h$

B 0

C h

D $2h$

E h^3

Solution:

By the definition, $h \otimes h = h^3 - h$. Then

$$h \otimes (h^3 - h) = h^3 - (h^3 - h) = h.$$

Thus, the correct answer is **C**.

3. The ratio of Mary's age to Alice's age is $3 : 5$. Alice is 30 years old. How old is Mary?

A 15

B 18

C 20

D 24

E 50

Solution:

Mary's age is $\frac{3}{5}$ of Alice's, so Mary is $\frac{3}{5} \cdot 30 = 18$ years old.

Thus, the correct answer is **B**.

4. A digital watch displays hours and minutes with AM and PM. What is the largest possible sum of the digits in the display?

A 17

B 19

C 21

D 22

E 23

Solution:

The two minutes digits sum to at most $5 + 9 = 14$, at 59 minutes past the hour. For the hour, a single digit 9 gives digit sum 9, which beats any two-digit hour $(10, 11, 12$ give at most $1 + 2 = 3)$.

The largest total is $14 + 9 = 23$, occurring at $9{:}59$.

Thus, the correct answer is **E**.

5. Doug and Dave shared a pizza with 8 equally-sized slices. Doug wanted a plain pizza, but Dave wanted anchovies on half of the pizza. The cost of a plain pizza was $\$8$, and there was an additional cost of $\$2$ for putting anchovies on one half. Dave ate all the slices of anchovy pizza and one plain slice. Doug ate the remainder. Each then paid for what he had eaten. How many more dollars did Dave pay than Doug?

A 1

B 2

C 3

D 4

E 5

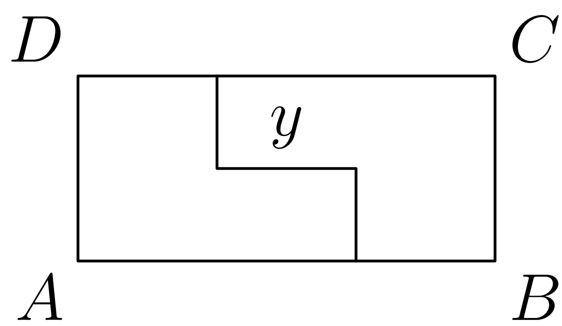
Solution:

Each plain slice costs $\$1.$ The $\$2$ anchovy charge is spread over the 4 anchovy slices, adding $\$0.50$ each, so an anchovy slice costs $\$1.50.$

Dave ate 4 anchovy slices and 1 plain slice: $4 \cdot 1.5 + 1 = \$7.$ Doug ate the 3 remaining plain slices: $\$3.$ Dave paid $7 - 3 = \$4$ more.

Thus, the correct answer is **D**.

6. The 8×18 rectangle $ABCD$ is cut into two congruent hexagons, as shown, in such a way that the two hexagons can be repositioned without overlap to form a square. What is y?



A 6

B 7

C 8

D 9

E 10

Solution:

The two hexagons form a square of area $8 \cdot 18 = 144$, so the square has side 12.

The staircase cut splits the width into three equal horizontal pieces of length y, which together span the full width: $y + y + y = 18$, so $y = 6$. (The two vertical steps each rise $12 - 8 = 4$, building the extra height of the square.)

Thus, the correct answer is **A**.

7. Mary is 20% older than Sally, and Sally is 40% younger than Danielle. The sum of their ages is 23.2 years. How old will Mary be on her next birthday?

A 7

B 8

C 9

D 10

E 11

Solution:

Let Danielle be x years old. Then Sally is $0.6x$ and Mary is $1.2(0.6x) = 0.72x$.

The sum $x + 0.6x + 0.72x = 2.32x = 23.2$ gives $x = 10$. So Mary is $0.72(10) = 7.2$ years old, and on her next birthday she will be 8.

Thus, the correct answer is **B**.

8. How many sets of two or more consecutive positive integers have a sum of 15?

A 1

B 2

C 3

D 4

E 5

Solution:

The sum of n consecutive integers equals n times their median. For a sum of $15 : n = 2$ gives $7 + 8, n = 3$ gives $4 + 5 + 6,$ and $n = 5$ gives $1 + 2 + 3 + 4 + 5.$

A run of four consecutive integers sums to an even number, and more than five terms already exceed $1 + 2 + 3 + 4 + 5 = 15.$ So there are 3 sets.

Thus, the correct answer is **C**.

9. Oscar buys 13 pencils and 3 erasers for $\$1.00$. A pencil costs more than an eraser, and both items cost a whole number of cents. What is the total cost, in cents, of one pencil and one eraser?

A 10

B 12

C 15

D 18

E 20

Solution:

Let p be a pencil's cost and s the cost of one pencil plus one eraser, in cents. Then

$$13p + 3e = 3s + 10p = 100,$$

so $3s$ is a multiple of 10 less than 100. Hence $s \in \{10, 20, 30\}$, with $p = 7, 4, 1$ respectively.

Since a pencil costs more than an eraser, $p > \frac{s}{2}$, which holds only for $s = 10$ (pencil 7, eraser 3). So one pencil and one eraser cost 10 cents.

Thus, the correct answer is **A**.

10. For how many real values of x is $\sqrt{120-\sqrt{x}}$ an integer?

A 3

B 6

C 9

D 10

E 11

Solution:

Let $k=\sqrt{120-\sqrt{x}}$ be an integer. Then $k \geq 0$ and $k^2=120-\sqrt{x} \leq 120$, so $0 \leq k \leq 10.$

Each such k gives $\sqrt{x}=120-k^2 \geq 0,$ hence a distinct value $x=(120-k^2)^2$. That is 11 values.

Thus, the correct answer is **E**.

11. Which of the following describes the graph of the equation $(x+y)^2 = x^2 + y^2$?

A the empty set

B one point

C two lines

D a circle

E the entire plane

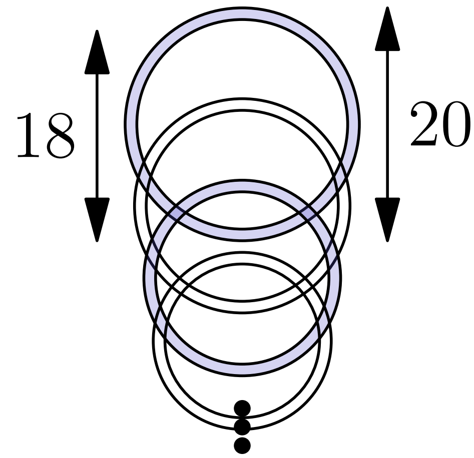
Solution:

Expanding, $x^2 + 2xy + y^2 = x^2 + y^2$, so $2xy = 0$, i.e. $xy = 0$.

This is the union of the two coordinate axes, a pair of lines.

Thus, the correct answer is **C**.

12. A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the other rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?



A 171

B 173

C 182

D 188

E 210

Solution:

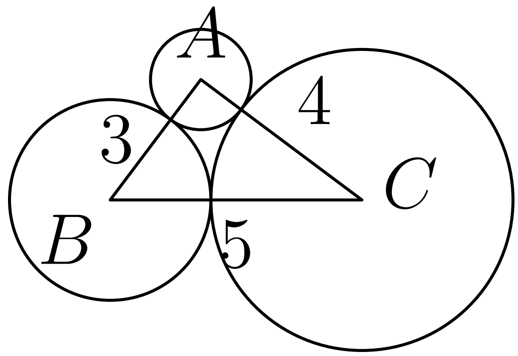
The top ring spans 20 cm. Each ring below overlaps the ring above by 2 cm (twice the 1 -cm thickness), so it adds its outside diameter minus 2.

The lower rings have outside diameters $19, 18, \ldots, 3$, contributing $17, 16, \ldots, 1$. Thus the total distance is

$$20 + (17 + 16 + \cdots + 1) = 20 + \frac{17 \cdot 18}{2} = 20 + 153 = 173 \text{ cm}.$$

Thus, the correct answer is **B**.

13. The vertices of a $3–4–5$ right triangle are the centers of three mutually externally tangent circles, as shown. What is the sum of the areas of these circles?



A 12π

B $\frac{25\pi}{2}$

C 13π

D $\frac{27\pi}{2}$

E 14π

Solution:

If r, s, t are the radii at the vertices, then $r + s = 3,\ r + t = 4,\ s + t = 5$. Adding all three gives $r + s + t = 6$, so $r = 1,\ s = 2,\ t = 3$.

The sum of the areas is $\pi(1^2 + 2^2 + 3^2) = 14\pi$.

Thus, the correct answer is **E**.

14. Two farmers agree that pigs are worth $\$300$ and that goats are worth $\$210$. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a $\$390$ debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

A $\$5$

B $\$10$

C $\$30$

D $\$90$

E $\$210$

Solution:

A debt D is resolvable if and only if $D = 300p + 210g = 30(10p + 7g)$ for integers p, g. Thus D is a multiple of $\gcd(300, 210) = 30$, so no smaller positive debt works.

A debt of $\$30$ is achievable since $30 = 300(-2) + 210(3)$, i.e. give 3 goats and receive 2 pigs in change.

Thus, the correct answer is **C**.

15. Suppose $\cos x = 0$ and $\cos(x+z) = \frac{1}{2}$. What is the smallest possible positive value of z?

A $\dfrac{\pi}{6}$

B $\dfrac{\pi}{3}$

C $\dfrac{\pi}{2}$

D $\dfrac{5\pi}{6}$

E $\dfrac{7\pi}{6}$

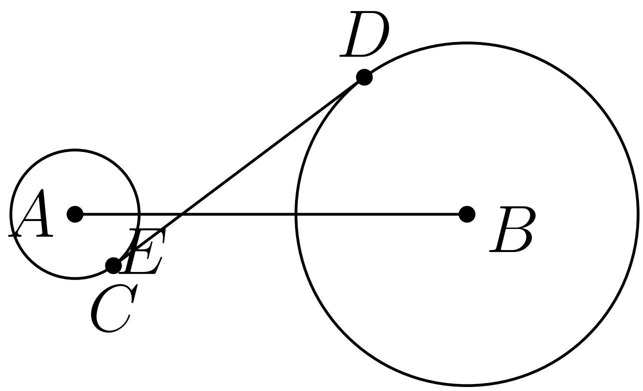
Solution:

Since $\cos x = 0$, we have $x = \frac{\pi}{2} + k\pi$. Since $\cos(x+z) = \frac{1}{2}$, we have $x + z = 2n\pi \pm \frac{\pi}{3}$.

Taking $x = -\frac{\pi}{2}$ and $x + z = -\frac{\pi}{3}$ gives $z = -\frac{\pi}{3} + \frac{\pi}{2} = \frac{\pi}{6}$, the smallest positive value.

Thus, the correct answer is **A**.

16. Circles with centers A and B have radii 3 and 8, respectively. A common internal tangent intersects the circles at C and D, respectively. Lines AB and CD intersect at E, and $AE = 5$. What is CD?



A 13

B $\frac{44}{3}$

C $\sqrt{221}$

D $\sqrt{255}$

E $\frac{55}{3}$

Solution:

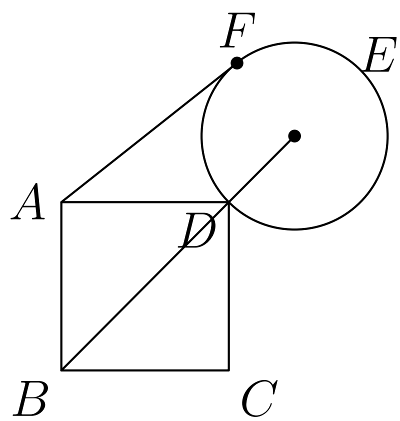
The radii satisfy $AC \perp CD$ and $BD \perp CD$. By the Pythagorean theorem, $CE = \sqrt{5^2 - 3^2} = 4$.

Since $\triangle ACE \sim \triangle BDE$, we get $\frac{DE}{CE} = \frac{BD}{AC} = \frac{8}{3}$, so $DE = 4 \cdot \frac{8}{3} = \frac{32}{3}$. Then

$$CD = CE + DE = 4 + \frac{32}{3} = \frac{44}{3}.$$

Thus, the correct answer is **B**.

17. Square $ABCD$ has side length s, a circle centered at E has radius r, and r and s are both rational. The circle passes through D, and D lies on $\overline{BE}$. Point F lies on the circle, on the same side of $\overline{BE}$ as A. Segment AF is tangent to the circle, and $AF = \sqrt{9+5\sqrt{2}}$. What is r/s?



A $\frac{1}{2}$

B $\frac{5}{9}$

C $\frac{3}{5}$

D $\frac{5}{3}$

E $\frac{9}{5}$

Solution:

Set $B=(0,0), C=(s,0), A=(0,s), D=(s,s)$, so that $E = \left(s+\frac{r}{\sqrt{2}},\ s+\frac{r}{\sqrt{2}}\right)$ lies on ray BD.

Since AF is tangent to the circle, $AF^2 = AE^2 - r^2$. Computing AE^2 and simplifying gives $9+5\sqrt{2} = s^2 + rs\sqrt{2}$.

Because r and s are rational, the rational and irrational parts match: $s^2 = 9$ and $rs = 5$. Thus $s = 3,\ r = \frac{5}{3}$, and $r/s = \frac{5}{9}$.

Thus, the correct answer is **B**.

18. The function f has the property that for each real number x in its domain, $1/x$ is also in its domain and

$$f(x) + f\left(\frac{1}{x}\right) = x.$$

What is the largest set of real numbers that can be in the domain of f?

A $\{x \mid x \neq 0\}$

B $\{x \mid x < 0\}$

C $\{x \mid x > 0\}$

D $\{x \mid x \neq -1 \text{ and } x \neq 0 \text{ and } x \neq 1\}$

E $\{-1, 1\}$

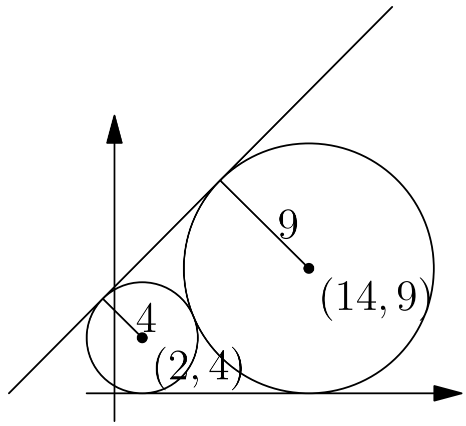
Solution:

Replacing x by $1/x$ gives $f\left(\frac{1}{x}\right) + f(x) = \frac{1}{x}$. Together with $f(x) + f\left(\frac{1}{x}\right) = x$, this requires $x = \frac{1}{x}$, so $x = \pm 1$.

Both values are consistent, with $f(1) = \frac{1}{2}$ and $f(-1) = -\frac{1}{2}$. So the largest possible domain is $\{-1, 1\}$.

Thus, the correct answer is **E**.

19. Circles with centers $(2,4)$ and $(14,9)$ have radii 4 and 9, respectively. The equation of a common external tangent to the circles can be written in the form $y = mx + b$ with $m > 0$. What is b?



A $\frac{908}{119}$

B $\frac{909}{119}$

C $\frac{130}{17}$

D $\frac{911}{119}$

E $\frac{912}{119}$

Solution:

Each circle's radius equals its center's y-coordinate, so both are tangent to the x-axis, which is a common external tangent. The two external tangents meet at the x-intercept of the line through the centers.

That line has slope $\frac{9-4}{14-2} = \frac{5}{12} = \tan\theta$ and passes through $(2,4)$, meeting the x-axis at $\left(-\frac{38}{5}, 0\right)$.

The other tangent makes angle 2θ with the x-axis, so its slope is

$$\tan 2\theta = \frac{2 \cdot \frac{5}{12}}{1 - \left(\frac{5}{12}\right)^2} = \frac{120}{119}.$$

Then $b = \frac{120}{119} \cdot \frac{38}{5} = \frac{912}{119}$.

Thus, the correct answer is **E**.

20. A bug starts at one vertex of a cube and moves along the edges of the cube according to the following rule. At each vertex the bug will choose to travel along one of the three edges emanating from that vertex. Each edge has equal probability of being chosen, and all choices are independent. What is the probability that after seven moves the bug will have visited every vertex exactly once?

A $\frac{1}{2187}$

B $\frac{1}{729}$

C $\frac{2}{243}$

D $\frac{1}{81}$

E $\frac{5}{243}$

Solution:

From the start there are 3^7 equally likely 7-move walks. Consider a walk visiting all 8 vertices: there are 3 choices for the first move and 2 for the second.

Labeling the first three vertices A, B, C, the bug must next move to one of two vertices, and in each case the remaining moves are forced. This gives $3 \cdot 2 \cdot 3 = 18$ such walks.

The probability is $\frac{18}{3^7} = \frac{18}{2187} = \frac{2}{243}$.

Thus, the correct answer is **C**.

21. Let

$$S_1 = \{(x, y) \mid \log_{10}(1 + x^2 + y^2) \le 1 + \log_{10}(x + y)\}$$

and

$$S_2 = \{(x, y) \mid \log_{10}(2 + x^2 + y^2) \le 2 + \log_{10}(x + y)\}.$$

What is the ratio of the area of S_2 to the area of S_1?

A 98

B 99

C 100

D 101

E 102

Solution:

For $j = 1, 2,$ the condition becomes $j + x^2 + y^2 \le 10^j(x + y)$, i.e.

$$\left(x - \frac{10^j}{2}\right)^2 + \left(y - \frac{10^j}{2}\right)^2 \le \frac{10^{2j}}{2} - j.$$

These are disks with squared radii $\frac{100}{2} - 1 = 49$ for S_1 and $\frac{10000}{2} - 2 = 4998$ for S_2. The area ratio is $\dfrac{4998}{49} = 102.$

Thus, the correct answer is **E**.

22. A circle of radius r is concentric with and outside a regular hexagon of side length 2. The probability that three entire sides of the hexagon are visible from a randomly chosen point on the circle is $1/2$. What is r?

A $2\sqrt{2}+2\sqrt{3}$

B $3\sqrt{3}+\sqrt{2}$

C $2\sqrt{6}+\sqrt{3}$

D $3\sqrt{2}+\sqrt{6}$

E $6\sqrt{2}-\sqrt{3}$

Solution:

Place the hexagon at the center of the circle. There are six congruent arcs from which three whole sides are visible; since the total probability is $\frac{1}{2}$, each arc measures 30°.

Take the arc centered at $(r, 0)$ with upper endpoint P, so $\angle POA = 15^\circ$. Then P lies on the line containing a side whose distance from the center is the apothem $\sqrt{3}$.

Hence $\sqrt{3} = r\sin 15^\circ = r \cdot \dfrac{\sqrt{6}-\sqrt{2}}{4}$, giving

$$r = \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} = 3\sqrt{2}+\sqrt{6}.$$

Thus, the correct answer is **D**.

23. Given a finite sequence $S = (a_1, a_2, \ldots, a_n)$ of n real numbers, let $A(S)$ be the sequence

$$\left(\frac{a_1 + a_2}{2}, \frac{a_2 + a_3}{2}, \ldots, \frac{a_{n-1} + a_n}{2}\right)$$

of $n - 1$ real numbers. Define $A^1(S) = A(S)$ and, for each integer m, $2 \le m \le n - 1$, define $A^m(S) = A(A^{m-1}(S))$. Suppose $x > 0$, and let $S = (1, x, x^2, \ldots, x^{100})$. If $A^{100}(S) = (1/2^{50})$, then what is x?

A $1 - \frac{\sqrt{2}}{2}$

B $\sqrt{2} - 1$

C $\frac{1}{2}$

D $2 - \sqrt{2}$

E $\frac{\sqrt{2}}{2}$

Solution:

Each application of A averages adjacent terms, so after 100 steps the single remaining term is

$$\frac{1}{2^{100}} \sum_{m=0}^{100} \binom{100}{m} x^m = \frac{(1+x)^{100}}{2^{100}}.$$

Setting this equal to $\frac{1}{2^{50}}$ gives $(1+x)^{100} = 2^{50}$, so $1 + x = 2^{1/2} = \sqrt{2}$. Since $x > 0$, we get $x = \sqrt{2} - 1$.

Thus, the correct answer is **B**.

24. The expression

$$(x+y+z)^{2006}+(x-y-z)^{2006}$$

is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

A 6018

B $671{,}676$

C $1{,}007{,}514$

D $1{,}008{,}016$

E $2{,}015{,}028$

Solution:

A term $x^a y^b z^c$ survives only when a is even, since terms with odd a cancel between the two expansions.

For each even a with $0 \leq a \leq 2006,$ the exponent b ranges over $2007-a$ values and $c=2006-a-b$ is then determined. Summing over even a :

$$(2007-0)+(2007-2)+\cdots+(2007-2006)=2007+2005+\cdots+1,$$

the sum of the first 1004 odd positive integers, which is $1004^2=1{,}008{,}016.$

Thus, the correct answer is **D**.

25. How many non-empty subsets S of $\{1, 2, 3, \ldots, 15\}$ have the following two properties?

(1) No two consecutive integers belong to S.

(2) If S contains k elements, then S contains no number less than k.

A 277

B 311

C 376

D 377

E 405

Solution:

By property (2), a valid k-element set is a k-subset of $\{k, k+1, \ldots, 15\}$ with no two consecutive elements.

Collapsing the gaps between chosen elements, these correspond bijectively to k-subsets of a $(17-2k)$-element set, counted by $\binom{17-2k}{k}$. This is nonzero only for $k \leq 5$, so the total is

$$\binom{15}{1} + \binom{13}{2} + \binom{11}{3} + \binom{9}{4} + \binom{7}{5} = 15 + 78 + 165 + 126 + 21 = 405.$$

Thus, the correct answer is **E**.

Problems: https://live.poshenloh.com/past-contests/amc12/2006A