2005 AMC 12B Solutions

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1. A scout troop buys 1000 candy bars at a price of five for $\$2$. They sell all the candy bars at a price of two for $\$1$. What was their profit, in dollars?

- A 100
- B 200
- C 300
- D 400
- E 500

Solution:

The troop buys $1000 \div 5 = 200$ groups of five bars, costing $200 \cdot 2 = 400$ dollars.

They sell $1000 \div 2 = 500$ pairs of bars, earning $500 \cdot 1 = 500$ dollars.

The profit is $500 - 400 = 100$ dollars.

Thus, the correct answer is **A**.

2. A positive number x has the property that $x\%$ of x is 4. What is x?

A 2

B 4

C 10

D 20

E 40

Solution:

The statement translates to

$$\frac{x}{100} \cdot x = 4,$$

so $x^2 = 400.$

Since x is positive, $x = 20.$

Thus, the correct answer is **D**.

3. Brianna is using part of the money she earned on her weekend job to buy several equally-priced CDs. She used one fifth of her money to buy one third of the CDs. What fraction of her money will she have left after she buys all the CDs?

A $\frac{1}{5}$

B $\frac{1}{3}$

C $\frac{2}{5}$

D $\frac{2}{3}$

E $\frac{4}{5}$

Solution:

Buying all the CDs costs three times what one third of them cost, namely $3 \cdot \frac{1}{5} = \frac{3}{5}$ of her money.

She has $1 - \frac{3}{5} = \frac{2}{5}$ of her money left.

Thus, the correct answer is **C**.

4. At the beginning of the school year, Lisa's goal was to earn an A on at least 80% of her 50 quizzes for the year. She earned an A on 22 of the first 30 quizzes. If she is to achieve her goal, on at most how many of the remaining quizzes can she earn a grade lower than an A?

A 1

B 2

C 3

D 4

E 5

Solution:

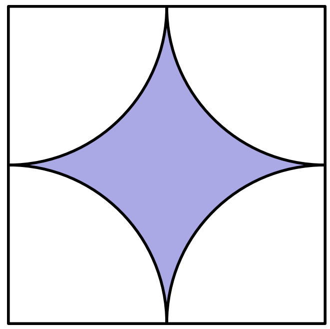
Lisa needs an A on at least $0.8 \cdot 50 = 40$ quizzes.

She has 22 already, so she needs $40 - 22 = 18$ more of the remaining 20 quizzes.

She can earn a lower grade on at most $20 - 18 = 2$ of them.

Thus, the correct answer is **B**.

5. An 8-foot by 10-foot floor is tiled with square tiles of size 1 foot by 1 foot. Each tile has a pattern consisting of four white quarter circles of radius $\frac{1}{2}$ foot centered at each corner of the tile. The remaining portion of the tile is shaded. How many square feet of the floor are shaded?



A $80 - 20\pi$

B $60 - 10\pi$

C $80 - 10\pi$

D $60 + 10\pi$

E $80 + 10\pi$

Solution:

The four quarter circles in a tile together form one full circle of radius $\frac{1}{2}$, with area $\pi\left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$.

So each tile has shaded area $1 - \frac{\pi}{4}$ square feet.

There are $8 \cdot 10 = 80$ tiles, so the total shaded area is

$$80\left(1 - \frac{\pi}{4}\right) = 80 - 20\pi.$$

Thus, the correct answer is **A**.

6. In $\triangle ABC$, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is BD?

A 3

B $2\sqrt{3}$

C 4

D 5

E $4\sqrt{2}$

Solution:

Let H be the foot of the altitude from C to line AB. Since $\triangle ABC$ is isosceles with $AC = BC$, H is the midpoint of AB, so $AH = HB = 1$.

Then $CH^2 = 7^2 - 1^2 = 48$. Applying the Pythagorean Theorem to $\triangle CHD$ with $HD = HB + BD = 1 + BD$ gives

$$8^2 = 48 + (1 + BD)^2,$$

so $(1 + BD)^2 = 16$.

Therefore $1 + BD = 4$, which means $BD = 3$.

Thus, the correct answer is **A**.

7. What is the area enclosed by the graph of $|3x| + |4y| = 12$?

A 6

B 12

C 16

D 24

E 25

Solution:

Setting $y = 0$ gives $|3x| = 12$, so $x = \pm 4$. Setting $x = 0$ gives $|4y| = 12$, so $y = \pm 3$.

The graph is a rhombus with vertices $(\pm 4, 0)$ and $(0, \pm 3)$, so its diagonals have lengths 8 and 6.

Its area is $\frac{1}{2} \cdot 8 \cdot 6 = 24$.

Thus, the correct answer is **D**.

8. For how many values of a is it true that the line $y = x + a$ passes through the vertex of the parabola $y = x^2 + a^2$?

- A 0
- B 1
- C 2
- D 10
- E infinitely many

Solution:

The vertex of the parabola $y = x^2 + a^2$ is $(0, a^2)$.

The line $y = x + a$ passes through it exactly when $a^2 = 0 + a$, that is $a^2 - a = 0$.

This gives $a = 0$ or $a = 1$, so there are 2 values.

Thus, the correct answer is **C**.

9. On a certain math exam, 10% of the students got 70 points, 25% got 80 points, 20% got 85 points, 15% got 90 points, and the rest got 95 points. What is the difference between the mean and the median score on this exam?

A 0

B 1

C 2

D 4

E 5

Solution:

The percentage scoring 95 is $100 - 10 - 25 - 20 - 15 = 30.$

The mean is

$$0.10(70) + 0.25(80) + 0.20(85) + 0.15(90) + 0.30(95) = 86.$$

Cumulatively, 10% are below $80, 35\%$ are at or below $80,$ and 55% are at or below $85.$ The middle scores fall at $85,$ so the median is $85.$

The difference is $86 - 85 = 1.$

Thus, the correct answer is **B**.

10. The first term of a sequence is 2005. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2005th term of the sequence?

A 29

B 55

C 85

D 133

E 250

Solution:

The sequence begins $2005, 133, 55, 250, 133, \ldots$ since $2^3 + 0^3 + 0^3 + 5^3 = 133$, $1^3 + 3^3 + 3^3 = 55$, $5^3 + 5^3 = 250$, and $2^3 + 5^3 + 0^3 = 133$.

After the initial 2005, the terms cycle through $133, 55, 250$ with period 3.

Term n for $n \geq 2$ is the $((n-2) \bmod 3)$th entry of $133, 55, 250$. Since $2005 - 2 = 2003 \equiv 2 \pmod 3$, the 2005th term is 250.

Thus, the correct answer is **E**.

11. An envelope contains eight bills: 2 ones, 2 fives, 2 tens, and 2 twenties. Two bills are drawn at random without replacement. What is the probability that their sum is $\$20$ or more?

A $\frac{1}{4}$

B $\frac{2}{7}$

C $\frac{3}{7}$

D $\frac{1}{2}$

E $\frac{2}{3}$

Solution:

There are $\binom{8}{2} = 28$ equally likely pairs of bills.

The sum is $\$20$ or more in these cases: both twenties (1 way), one twenty with one of the six smaller bills ($2 \cdot 6 = 12$ ways), or both tens (1 way).

That is $1 + 12 + 1 = 14$ favorable pairs, so the probability is $\frac{14}{28} = \frac{1}{2}$.

Thus, the correct answer is **D**.

12. The quadratic equation $x^2 + mx + n = 0$ has roots that are twice those of $x^2 + px + m = 0$, and none of $m, n,$ and p is zero. What is the value of $\frac{n}{p}$?

A	1
B	2
C	4
D	8
E	16

Solution:

Let r_1 and r_2 be the roots of $x^2 + px + m = 0$, so $m = r_1 r_2$ and $p = -(r_1 + r_2)$.

The roots of $x^2 + mx + n = 0$ are $2r_1$ and $2r_2$, so $n = 4r_1 r_2$ and $m = -2(r_1 + r_2)$.

Then $n = 4m$ and $m = 2p$, which gives $p = \frac{m}{2}$, so

$$\frac{n}{p} = \frac{4m}{\frac{m}{2}} = 8.$$

Thus, the correct answer is **D**.

13. Suppose that $4^{x_1} = 5, 5^{x_2} = 6, 6^{x_3} = 7, \ldots, 127^{x_{124}} = 128$. What is $x_1 x_2 \cdots x_{124}$?

A 2

B $\frac{5}{2}$

C 3

D $\frac{7}{2}$

E 4

Solution:

From $4^{x_1} = 5$ we get $x_1 = \log_4 5$, and in general $x_k = \log_{k+3}(k+4)$.

The product telescopes:

$$x_1 x_2 \cdots x_{124} = \log_4 5 \cdot \log_5 6 \cdots \log_{127} 128 = \log_4 128.$$

Since $128 = 2^7$ and $4 = 2^2$, this equals $\dfrac{7 \log 2}{2 \log 2} = \dfrac{7}{2}$.

Thus, the correct answer is **D**.

14. A circle having center $(0, k)$, with $k > 6$, is tangent to the lines $y = x, y = -x$ and $y = 6$. What is the radius of this circle?

A $6\sqrt{2} - 6$

B 6

C $6\sqrt{2}$

D 12

E $6 + 6\sqrt{2}$

Solution:

Since the circle is tangent to $y = 6$ and its center $(0, k)$ is above that line, the radius is $r = k - 6$.

The distance from $(0, k)$ to the line $x - y = 0$ is $\frac{|0 - k|}{\sqrt{2}} = \frac{k}{\sqrt{2}}$, and this must also equal r.

Setting $\frac{k}{\sqrt{2}} = k - 6$ gives $k = \frac{6\sqrt{2}}{\sqrt{2} - 1} = 6\sqrt{2}\,(\sqrt{2} + 1) = 12 + 6\sqrt{2}$.

Then $r = k - 6 = 6 + 6\sqrt{2}$.

Thus, the correct answer is **E**.

15. The sum of four two-digit numbers is 221. None of the eight digits is 0 and no two of them are the same. Which of the following is not included among the eight digits?

A 1

B 2

C 3

D 4

E 5

Solution:

The eight digits are distinct and chosen from 1 through 9, whose total is 45. So the eight used digits sum to between $45-9=36$ and $45-1=44$.

Let the four units digits sum to U and the four tens digits sum to T. Then $10T+U=221$, so U ends in 1. Since $1+2+3+4=10 \leq U \leq 6+7+8+9=30$, we have $U=11$ or $U=21$.

If $U=11$, then $10T=210$, so $T=21$ and the eight digits sum to 32, which is below 36. So $U=21$, giving $T=20$ and total 41.

The missing digit is $45-41=4$. For example, $13+25+86+97=221$.

Thus, the correct answer is **D**.

16. Eight spheres of radius 1, one per octant, are each tangent to the coordinate planes. What is the radius of the smallest sphere, centered at the origin, that contains these eight spheres?

A $\sqrt{2}$

B $\sqrt{3}$

C $1+\sqrt{2}$

D $1+\sqrt{3}$

E 3

Solution:

A sphere of radius 1 tangent to the three coordinate planes in one octant has its center at a point like $(1,1,1)$, at distance

$$\sqrt{1^2+1^2+1^2}=\sqrt{3}$$

from the origin.

The farthest point of that sphere from the origin is at distance $\sqrt{3}+1$, so the containing sphere has radius $1+\sqrt{3}$.

Thus, the correct answer is **D**.

17. How many distinct four-tuples (a, b, c, d) of rational numbers are there with

$$a \log_{10} 2 + b \log_{10} 3 + c \log_{10} 5 + d \log_{10} 7 = 2005?$$

- A 0
- B 1
- C 17
- D 2004
- E infinitely many

Solution:

The equation is equivalent to $\log_{10}\left(2^a 3^b 5^c 7^d\right) = 2005$, so

$$2^a 3^b 5^c 7^d = 10^{2005} = 2^{2005} \cdot 5^{2005}.$$

Clearing the denominators of a, b, c, d with a common integer multiplier and using the uniqueness of prime factorization, the exponents must match: $a = 2005, b = 0, c = 2005$, and $d = 0$.

So there is exactly 1 such four-tuple.

Thus, the correct answer is **B**.

18. Let $A(2, 2)$ and $B(7, 7)$ be points in the plane. Define R as the region in the first quadrant consisting of those points C such that $\triangle ABC$ is an acute triangle. What is the closest integer to the area of the region R?

A 25

B 39

C 51

D 60

E 80

Solution:

Line AB has slope 1. For $\angle A$ to be acute, C must lie beyond the line through A perpendicular to AB; in the first quadrant that line runs between $P(4, 0)$ and $Q(0, 4)$. For $\angle B$ to be acute, C must lie before the line through B perpendicular to AB, between $S(14, 0)$ and $T(0, 14)$.

For $\angle C$ to be acute, C must lie outside the circle U with diameter AB, whose radius is $\frac{AB}{2} = \frac{5\sqrt{2}}{2}$.

The region is the large right triangle OST minus the small right triangle OPQ minus the semicircle-equivalent area of U inside the strip:

$$\frac{1}{2} \cdot 14^2 - \frac{1}{2} \cdot 4^2 - \pi \left(\frac{5\sqrt{2}}{2}\right)^2 = 98 - 8 - \frac{25\pi}{2} = 90 - \frac{25\pi}{2} \approx 51.$$

Thus, the correct answer is **C**.

19. Let x and y be two-digit integers such that y is obtained by reversing the digits of x. The integers x and y satisfy $x^2 - y^2 = m^2$ for some positive integer m. What is $x + y + m$?

A 88

B 112

C 116

D 144

E 154

Solution:

Let $x = 10a + b$ and $y = 10b + a$ with $a > b$. Then

$$x^2 - y^2 = (10a+b)^2 - (10b+a)^2 = 99(a^2 - b^2) = 99(a+b)(a-b).$$

Since $99 = 9 \cdot 11$, for this to be a perfect square we need $11 \mid (a+b)(a-b)$. As $a + b \leq 17$ and $a - b \leq 8$, the only multiple of 11 available is $a + b = 11$.

Then $x^2 - y^2 = 9 \cdot 11^2(a-b)$, which is a perfect square exactly when $a - b$ is a perfect square. Taking $a - b = 1$ with $a + b = 11$ gives $(a, b) = (6, 5)$.

So $x = 65, y = 56$, and $m = \sqrt{65^2 - 56^2} = \sqrt{1089} = 33$. Thus $x + y + m = 65 + 56 + 33 = 154$.

Thus, the correct answer is **E**.

20. Let a, b, c, d, e, f, g and h be distinct elements in the set

$$\{-7, -5, -3, -2, 2, 4, 6, 13\}.$$

What is the minimum possible value of

$$(a+b+c+d)^2 + (e+f+g+h)^2?$$

- A 30
- B 32
- C 34
- D 40
- E 50

Solution:

The elements sum to 8. If $a+b+c+d = x$, then $e+f+g+h = 8-x$, so

$$x^2 + (8-x)^2 = 2(x-4)^2 + 32.$$

This is minimized when $x = 4$, giving 32. But 13 must lie in one group, and no three of the remaining elements add with 13 to make 4 (that would need three of them summing to -9, which is impossible here). So $x = 4$ is unattainable and $(x-4)^2 \geq 1$.

The minimum is $2(1) + 32 = 34$, achieved for instance by $\{-7, -5, 2, 13\}$ (sum 3) and $\{-3, -2, 4, 6\}$ (sum 5).

Thus, the correct answer is **C**.

21. A positive integer n has 60 divisors and $7n$ has 80 divisors. What is the greatest integer k such that 7^k divides n?

A 0

B 1

C 2

D 3

E 4

Solution:

Write $n = 7^k Q$ where $7 \nmid Q$, and let d be the number of divisors of Q. Then n has $(k+1)d = 60$ divisors and $7n = 7^{k+1}Q$ has $(k+2)d = 80$ divisors.

Dividing, $\dfrac{k+2}{k+1} = \dfrac{80}{60} = \dfrac{4}{3}$, so $3(k+2) = 4(k+1)$, giving $k = 2$.

Thus, the correct answer is **C**.

22. A sequence of complex numbers $z_0, z_1, z_2, \ldots$ is defined by the rule

$$z_{n+1} = \frac{iz_n}{\overline{z_n}},$$

where $\overline{z_n}$ is the complex conjugate of z_n and $i^2 = -1$. Suppose that $|z_0| = 1$ and $z_{2005} = 1$. How many possible values are there for z_0?

A 1

B 2

C 4

D 2005

E 2^{2005}

Solution:

Because $|z_0| = 1$, every $|z_n| = 1$, so $\overline{z_n} = \dfrac{1}{z_n}$ and

$$z_{n+1} = \frac{iz_n}{\overline{z_n}} = iz_n^2.$$

Iterating, $z_1 = iz_0^2$, $z_2 = i(iz_0^2)^2 = -iz_0^4$, and in general for $n \geq 2$, $z_n = (\text{constant of modulus 1}) \cdot z_0^{2^n}$.

The condition $z_{2005} = 1$ becomes an equation of the form $z_0^{2^{2005}} = c$ for a fixed constant c with $|c| = 1$. Every nonzero complex equation $z_0^N = c$ has exactly N distinct solutions, all on the unit circle.

Here $N = 2^{2005}$, so there are 2^{2005} possible values for z_0.

Thus, the correct answer is **E**.

23. Let S be the set of ordered triples (x, y, z) of real numbers for which

$$\log_{10}(x+y) = z \quad \text{and} \quad \log_{10}(x^2+y^2) = z+1.$$

There are real numbers a and b such that for all ordered triples (x, y, z) in S we have $x^3 + y^3 = a \cdot 10^{3z} + b \cdot 10^{2z}$. What is the value of $a + b$?

A $\frac{15}{2}$

B $\frac{29}{2}$

C 15

D $\frac{39}{2}$

E 24

Solution:

The conditions give $x + y = 10^z$ and $x^2 + y^2 = 10 \cdot 10^z$. Then

$$2xy = (x+y)^2 - (x^2+y^2) = 10^{2z} - 10 \cdot 10^z,$$

so $xy = \frac{1}{2}\left(10^{2z} - 10 \cdot 10^z\right)$.

Using $x^3 + y^3 = (x+y)^3 - 3xy(x+y)$,

$$x^3 + y^3 = 10^{3z} - \frac{3}{2}\left(10^{2z} - 10 \cdot 10^z\right)10^z = -\frac{1}{2} \cdot 10^{3z} + 15 \cdot 10^{2z}.$$

So $a = -\frac{1}{2}$ and $b = 15$, giving $a + b = \frac{29}{2}$.

Thus, the correct answer is **B**.

24. All three vertices of an equilateral triangle are on the parabola $y = x^2$, and one of its sides has a slope of 2. The x-coordinates of the three vertices have a sum of $\frac{m}{n}$, where m and n are relatively prime positive integers. What is the value of $m + n$?

A 14

B 15

C 16

D 17

E 18

Solution:

For vertices $(a, a^2), (b, b^2), (c, c^2)$, the slope of a side is $\frac{b^2 - a^2}{b - a} = a + b$. Adding the three side slopes,

$$(a + b) + (b + c) + (c + a) = 2(a + b + c) = 2 \cdot \frac{m}{n}.$$

One side has slope $2 = \tan\theta$. Because the triangle is equilateral, its sides make angles θ and $\theta \pm 60^\circ$, so the other two slopes are

$$\tan(\theta \pm 60^\circ) = \frac{2 \pm \sqrt{3}}{1 \mp 2\sqrt{3}} = -\frac{8 \pm 5\sqrt{3}}{11}.$$

The sum of the three slopes is $2 - \frac{8 + 5\sqrt{3}}{11} - \frac{8 - 5\sqrt{3}}{11} = \frac{22 - 16}{11} = \frac{6}{11}$.

Thus $a + b + c = \frac{1}{2} \cdot \frac{6}{11} = \frac{3}{11}$, so $m + n = 3 + 11 = 14$.

Thus, the correct answer is **A**.

25. Six ants simultaneously stand on the six vertices of a regular octahedron, with each ant at a different vertex. Simultaneously and independently, each ant moves from its vertex to one of the four adjacent vertices, each with equal probability. What is the probability that no two ants arrive at the same vertex?

A $\frac{5}{256}$

B $\frac{21}{1024}$

C $\frac{11}{512}$

D $\frac{23}{1024}$

E $\frac{3}{128}$

Solution:

There are 4^6 equally likely combinations of moves. Label the vertices A, B, C, A', B', C', where primed vertices are opposite the corresponding unprimed ones. An ant cannot move to its own vertex or the opposite one, so a valid outcome is a permutation f with $f(A) \notin \{A, A'\}$, and similarly for each pair.

There are $4 \cdot 3 = 12$ ordered choices for $(f(A), f(A'))$. Of these, $f(A)$ and $f(A')$ are opposite in 4 cases and adjacent in 8.

If $f(A), f(A')$ are opposite, say B, B', then $\{f(C), f(C')\} = \{A, A'\}$ and $\{f(B), f(B')\} = \{C, C'\}$, giving $4 \cdot 2 \cdot 2 = 16$ valid combinations.

If $f(A), f(A')$ are adjacent, say B, C, then one of $f(B), f(B')$ must be C' and there are 4 ordered choices for $(f(B), f(B'))$, each leaving 2 for $(f(C), f(C'))$: that is $8 \cdot 4 \cdot 2 = 64$ valid combinations.

Hence the probability is

$$\frac{16+64}{4^6} = \frac{80}{4096} = \frac{5}{256}.$$

Thus, the correct answer is **A**.

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