2005 AMC 12A Solutions

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1. Two is 10% of x and 20% of y. What is $x - y$?

A 1

B 2

C 5

D 10

E 20

Solution:

From $0.1x = 2$ we get $x = 20$, and from $0.2y = 2$ we get $y = 10$.

Therefore $x - y = 20 - 10 = 10$.

Thus, the correct answer is **D**.

2. The equations $2x + 7 = 3$ and $bx - 10 = -2$ have the same solution for x. What is the value of b?

- A -8
- **B** -4
- C -2
- D 4
- E 8

Solution:

Solving $2x + 7 = 3$ gives $x = -2.$

Substituting into the second equation, $-2b - 10 = -2,$ so $-2b = 8$ and $b = -4.$

Thus, the correct answer is **B**.

3. A rectangle with a diagonal of length x is twice as long as it is wide. What is the area of the rectangle?

A $\frac{1}{4}x^2$

B $\frac{2}{5}x^2$

C $\frac{1}{2}x^2$

D x^2

E $\frac{3}{2}x^2$

Solution:

Let the width be w. Then the length is $2w$, and the diagonal gives

$$x^2 = w^2 + (2w)^2 = 5w^2.$$

The area is $w \cdot 2w = 2w^2 = \frac{2}{5}x^2$.

Thus, the correct answer is **B**.

4. A store normally sells windows at $\$100$ each. This week the store is offering one free window for each purchase of four. Dave needs seven windows and Doug needs eight windows. How many dollars will they save if they purchase the windows together rather than separately?

A 100

B 200

C 300

D 400

E 500

Solution:

Buying separately, Dave gets 7 windows by paying for 6 ($\$600$), and Doug gets 8 by paying for 7 ($\$700$), for a total of $\$1300$.

Buying together, they need 15 windows: paying for 12 yields 3 free, for a cost of $\$1200$.

The savings are $\$1300 - \$1200 = \$100$.

Thus, the correct answer is **A**.

5. The average (mean) of 20 numbers is 30, and the average of 30 other numbers is 20. What is the average of all 50 numbers?

A 23

B 24

C 25

D 26

E 27

Solution:

The total of all 50 numbers is $20 \cdot 30 + 30 \cdot 20 = 600 + 600 = 1200.$

The average is $1200 \div 50 = 24.$

Thus, the correct answer is **B**.

6. Josh and Mike live 13 miles apart. Yesterday Josh started to ride his bicycle toward Mike's house. A little later Mike started to ride his bicycle toward Josh's house. When they met, Josh had ridden for twice the length of time as Mike and at four-fifths of Mike's rate. How many miles had Mike ridden when they met?

- A 4
- B 5
- C 6
- D 7
- E 8

Solution:

Since distance is rate times time, Josh rode $\frac{4}{5} \cdot 2 = \frac{8}{5}$ as far as Mike.

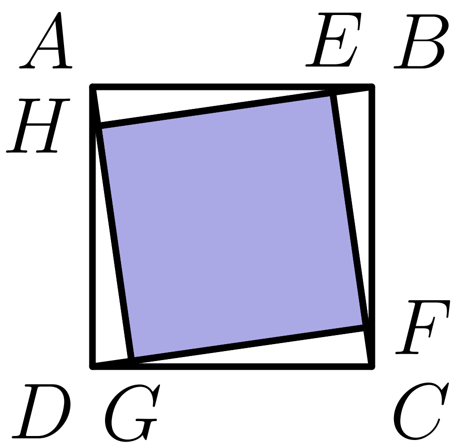
Let m be the miles Mike rode. Then

$$13 = m + \frac{8}{5}m = \frac{13}{5}m,$$

so $m = 5.$

Thus, the correct answer is **B**.

7. Square $EFGH$ is inside square $ABCD$ so that each side of $EFGH$ can be extended to pass through a vertex of $ABCD$. Square $ABCD$ has side length $\sqrt{50}$, E is between B and H, and $BE = 1$. What is the area of the inner square $EFGH$?



A 25

B 32

C 36

D 40

E 42

Solution:

By the symmetry of the figure, triangles ABH, BCE, CDF, and DAG are congruent right triangles. Hence

$$BH = CE = \sqrt{BC^2 - BE^2} = \sqrt{50 - 1} = 7.$$

Since E lies between B and H, the side of the inner square is $EH = BH - BE = 7 - 1 = 6.$

Therefore the area of $EFGH$ is $6^2 = 36.$

Thus, the correct answer is **C**.

8. Let A, M, and C be digits with

$$(100A + 10M + C)(A + M + C) = 2005.$$

What is A?

A 1

B 2

C 3

D 4

E 5

Solution:

Since $A + M + C \leq 9 + 9 + 9 = 27$, and $2005 = 5 \cdot 401$, the digit sum must be the smaller factor:

$$100A + 10M + C = 401, \quad A + M + C = 5.$$

Reading off the digits, $A = 4, M = 0$, and $C = 1$.

Thus, the correct answer is **D**.

9. There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. What is the sum of those values of a?

A -16

B -8

C 0

D 8

E 20

Solution:

The equation is $4x^2 + (a+8)x + 9 = 0$. It has one solution when the discriminant vanishes:

$$(a+8)^2 - 4 \cdot 4 \cdot 9 = 0,$$

so $(a+8)^2 = 144$ and $a + 8 = \pm 12$.

Thus $a = 4$ or $a = -20$, and their sum is -16.

Thus, the correct answer is **A**.

10. A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n?

A 3

B 4

C 5

D 6

E 7

Solution:

The n^3 unit cubes have $6n^3$ faces total. The red faces are exactly the surface of the original cube, $6n^2$ of them.

Setting the red fraction to one-fourth,

$$\frac{6n^2}{6n^3} = \frac{1}{n} = \frac{1}{4},$$

so $n = 4.$

Thus, the correct answer is **B**.

11. How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

A 41

B 42

C 43

D 44

E 45

Solution:

The middle digit is an integer only when the first and last digits are both odd or both even. Each such pair determines the middle digit uniquely.

There are $5 \cdot 5 = 25$ odd-odd choices for the first and last digits. For even-even, the first digit cannot be 0, giving $4 \cdot 5 = 20$ choices.

The total is $25 + 20 = 45.$

Thus, the correct answer is **E**.

12. A line passes through $A(1,1)$ and $B(100,1000)$. How many other points with integer coordinates are on the line and strictly between A and B?

- A 0
- B 2
- C 3
- D 8
- E 9

Solution:

The slope is

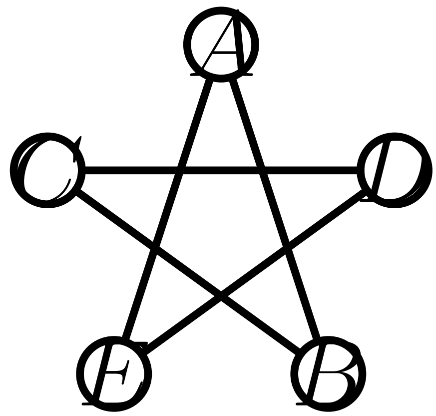
$$\frac{1000-1}{100-1} = \frac{999}{99} = \frac{111}{11}.$$

So every point on the line has the form $(1+11t,\ 1+111t)$, which is a lattice point exactly when t is an integer. The point is strictly between A and B when $0 < t < 9$.

There are 8 such integers t, giving 8 lattice points.

Thus, the correct answer is **D**.

13. In the five-sided star shown, the letters $A, B, C, D,$ and E are replaced by the numbers $3, 5, 6, 7,$ and $9,$ although not necessarily in that order. The sums of the numbers at the ends of the line segments $AB, BC, CD, DE,$ and EA form an arithmetic sequence, although not necessarily in that order. What is the middle term of the arithmetic sequence?



A 9

B 10

C 11

D 12

E 13

Solution:

Every number appears as an endpoint of exactly two of the five segments, so the total of the five sums is

$$2(3 + 5 + 6 + 7 + 9) = 60.$$

The middle term of a five-term arithmetic sequence is its mean, namely $60 \div 5 = 12.$

Thus, the correct answer is **D**.

14. On a standard die one of the dots is removed at random with each dot equally likely to be chosen. The die is then rolled. What is the probability that the top face has an odd number of dots?

A $\frac{5}{11}$

B $\frac{10}{21}$

C $\frac{1}{2}$

D $\frac{11}{21}$

E $\frac{6}{11}$

Solution:

The die has 21 dots, so a dot is removed from the face with n dots with probability $\frac{n}{21}$.

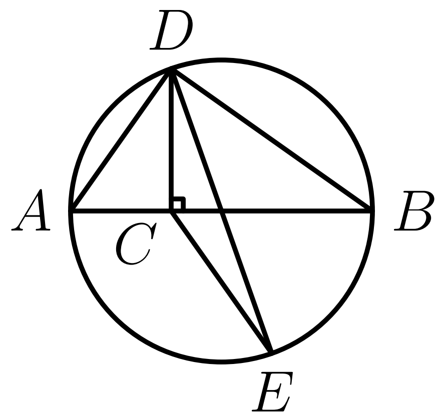
If a dot is removed from an odd face, the top is odd with probability $\frac{1}{3}$ (any of the three odd faces on top); if from an even face, the top is odd with probability $\frac{2}{3}$. The removed dot lies on an odd face with probability $\frac{1+3+5}{21}$ and an even face with probability $\frac{2+4+6}{21}$.

Hence the answer is

$$\frac{1}{3}\cdot\frac{9}{21}+\frac{2}{3}\cdot\frac{12}{21}=\frac{33}{63}=\frac{11}{21}.$$

Thus, the correct answer is **D**.

15. Let AB be a diameter of a circle and C be a point on AB with $2 \cdot AC = BC$. Let D and E be points on the circle such that $DC \perp AB$ and DE is a second diameter. What is the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$?



A $\frac{1}{6}$

B $\frac{1}{4}$

C $\frac{1}{3}$

D $\frac{1}{2}$

E $\frac{2}{3}$

Solution:

Let O be the center. Since $2 \cdot AC = BC$, we have $AC = \dfrac{AB}{3}$, and $AO = \dfrac{AB}{2}$, so

$$CO = AO - AC = \frac{AB}{2} - \frac{AB}{3} = \frac{AB}{6}.$$

Triangles DCO and DAB share the same altitude from D to line AB, so

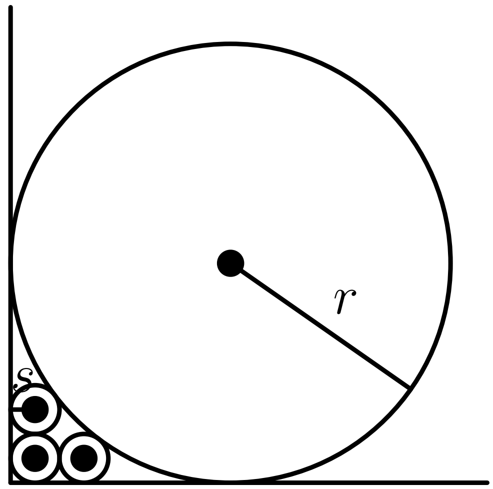
$$\frac{[DCO]}{[DAB]} = \frac{CO}{AB} = \frac{1}{6}.$$

Because O is the midpoint of DE, triangles DCO and ECO have equal areas, so

$$[DCE] = 2\,[DCO] = \frac{2}{6}[DAB] = \frac{1}{3}[DAB].$$

Thus, the correct answer is **C**.

16. Three circles of radius s are drawn in the first quadrant of the xy-plane. The first circle is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis. A circle of radius $r > s$ is tangent to both axes and to the second and third circles. What is r/s?



A 5

B 6

C 8

D 9

E 10

Solution:

Put the big circle's center at (r, r) and the second small circle's center at $(3s, s)$. They are externally tangent, so the distance between centers is $r + s$.

The horizontal and vertical gaps are $r - 3s$ and $r - s$, so

$$(r+s)^2 = (r-3s)^2 + (r-s)^2.$$

Expanding gives $0 = r^2 - 10rs + 9s^2 = (r-9s)(r-s)$. Since $r \neq s$, we get $r = 9s$, so $r/s = 9$.

Thus, the correct answer is **D**.

17. A unit cube is cut twice to form three triangular prisms, two of which are congruent, as shown in Figure 1. The cube is then cut in the same manner along the dashed lines shown in Figure 2. This creates nine pieces. What is the volume of the piece that contains vertex W?

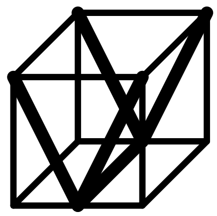


Figure 1

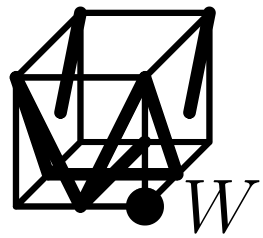


Figure 2

A $\frac{1}{12}$

B $\frac{1}{9}$

C $\frac{1}{8}$

D $\frac{1}{6}$

E $\frac{1}{4}$

Solution:

The two sets of cuts each run from a top edge down to the midline of the bottom face. Near W they carve out a pyramid whose apex is the top vertex directly above W.

Its base is a square of side $\frac{1}{2}$ (a quarter of the bottom face) and its altitude is the full height 1. Therefore the volume is

$$\frac{1}{3}\left(\frac{1}{2}\right)^2(1) = \frac{1}{12}.$$

Thus, the correct answer is **A**.

18. Call a number "prime-looking" if it is composite but not divisible by $2, 3,$ or $5.$ The three smallest prime-looking numbers are $49, 77,$ and $91.$ There are 168 prime numbers less than $1000.$ How many prime-looking numbers are there less than $1000?$

A 100

B 102

C 104

D 106

E 108

Solution:

Among the 999 numbers from 1 to $999,$ inclusion-exclusion gives

$$499 + 333 + 199 - 166 - 99 - 66 + 33 = 733$$

that are divisible by $2, 3,$ or $5.$

That leaves $999 - 733 = 266$ numbers coprime to $2, 3, 5.$ Of these, 165 are primes (the 168 primes minus $2, 3, 5$), and 1 is neither prime nor composite.

The remaining $266 - 165 - 1 = 100$ numbers are prime-looking.

Thus, the correct answer is **A**.

19. A faulty car odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. For example, after traveling one mile the odometer changed from 000039 to 000050. If the odometer now reads 002005, how many miles has the car actually traveled?

A 1404

B 1462

C 1604

D 1605

E 1804

Solution:

Because the odometer never displays a 4, it uses only 9 symbols and counts in base 9, where its digits $5, 6, 7, 8, 9$ represent the base-9 digits $4, 5, 6, 7, 8$.

The reading 002005 therefore corresponds to 2004 in base 9, which equals

$$2 \cdot 9^3 + 4 = 2 \cdot 729 + 4 = 1462.$$

Thus, the correct answer is **B**.

20. For each x in $[0, 1]$, define

$$f(x) = \begin{cases} 2x, & \text{if } 0 \le x \le \frac{1}{2}, \\ 2 - 2x, & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

Let $f^{[2]}(x) = f(f(x))$, and $f^{[n+1]}(x) = f^{[n]}(f(x))$ for each integer $n \ge 2$. For how many values of x in $[0, 1]$ is $f^{[2005]}(x) = \frac{1}{2}$?

A 0

B 2005

C 4010

D 2005^2

E 2^{2005}

Solution:

Let $g(n)$ count the solutions of $f^{[n]}(x) = \frac{1}{2}$ in $[0, 1]$. Since f maps each of the two halves $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ onto all of $[0, 1]$, every solution of $f^{[n-1]}(y) = \frac{1}{2}$ comes from two values of x (one in each half).

The boundary value $x = \frac{1}{2}$ satisfies $f^{[n]}(\frac{1}{2}) = f^{[n-1]}(1) = 0 \ne \frac{1}{2}$, so no solutions are lost, giving $g(n) = 2\,g(n-1)$.

Since $g(1) = 2$, we conclude $g(2005) = 2^{2005}$.

Thus, the correct answer is **E**.

21. How many ordered triples of integers (a, b, c), with $a \geq 2, b \geq 1$, and $c \geq 0$, satisfy both $\log_a b = c^{2005}$ and $a + b + c = 2005$?

A 0

B 1

C 2

D 3

E 4

Solution:

The condition $\log_a b = c^{2005}$ means $b = a^{\left(c^{2005}\right)}$.

If $c \geq 2$, then $b = a^{\left(c^{2005}\right)} \geq 2^{\left(2^{2005}\right)}$, which vastly exceeds 2005, so $a + b + c = 2005$ is impossible.

For $c = 0 : b = a^0 = 1$, so $a + 1 + 0 = 2005$ gives $(a, b, c) = (2004, 1, 0)$. For $c = 1 : b = a^1 = a$, so $2a + 1 = 2005$ gives $(a, b, c) = (1002, 1002, 1)$.

There are 2 such triples.

Thus, the correct answer is **C**.

22. A rectangular box P is inscribed in a sphere of radius r. The surface area of P is 384, and the sum of the lengths of its 12 edges is 112. What is r?

A 8

B 10

C 12

D 14

E 16

Solution:

Let the dimensions be x, y, z. The 12 edges give $4(x+y+z) = 112$, so $x+y+z = 28$, and the surface area gives $2xy + 2yz + 2xz = 384$.

The space diagonal is a diameter of the sphere, so

$$(2r)^2 = x^2 + y^2 + z^2 = (x+y+z)^2 - (2xy + 2yz + 2xz) = 28^2 - 384 = 400.$$

Thus $2r = 20$ and $r = 10$.

Thus, the correct answer is **B**.

23. Two distinct numbers a and b are chosen randomly from the set $\{2, 2^2, 2^3, \ldots, 2^{25}\}$. What is the probability that $\log_a b$ is an integer?

A $\frac{2}{25}$

B $\frac{31}{300}$

C $\frac{13}{100}$

D $\frac{7}{50}$

E $\frac{1}{2}$

Solution:

Let $a = 2^j$ and $b = 2^k$. Then $\log_a b = \dfrac{k}{j}$, which is an integer exactly when $j \mid k$.

For each j, the number of valid $k \neq j$ in $\{1, \ldots, 25\}$ is $\left\lfloor \frac{25}{j} \right\rfloor - 1$. Summing over j gives

$$24 + 11 + 7 + 5 + 4 + 3 + 2 + 2 + 4 \cdot 1 = 62$$

ordered pairs (a, b).

Since there are $25 \cdot 24 = 600$ ordered pairs of distinct elements, the probability is $\dfrac{62}{600} = \dfrac{31}{300}$.

Thus, the correct answer is **B**.

24. Let $P(x) = (x-1)(x-2)(x-3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P(Q(x)) = P(x) \cdot R(x)$?

A 19

B 22

C 24

D 27

E 32

Solution:

Since $P(x)R(x)$ has degree 6 and $P(Q(x))$ has degree $3 \deg Q$, we need $\deg Q = 2$. A quadratic Q is determined by the ordered triple $(Q(1), Q(2), Q(3))$.

At $x = 1, 2, 3$ the right side vanishes, so $P(Q(x)) = 0$, forcing each of $Q(1), Q(2), Q(3)$ into $\{1, 2, 3\}$. That gives 27 triples.

Five of them give a polynomial of degree less than 2 : the constants from $(1,1,1), (2,2,2), (3,3,3)$ and the linear $Q(x) = x$ from $(1,2,3)$ and $Q(x) = 4 - x$ from $(3,2,1)$. The other $27 - 5 = 22$ triples are non-collinear and yield genuine quadratics.

Thus, the correct answer is **B**.

25. Let S be the set of all points with coordinates (x, y, z), where $x, y,$ and z are each chosen from the set $\{0, 1, 2\}$. How many equilateral triangles have all their vertices in S?

A 72

B 76

C 80

D 84

E 88

Solution:

The three equal sides of such a triangle must all have the same length. Checking the possible squared lengths in the $3 \times 3 \times 3$ grid, only three families of side occur.

Face diagonals of a unit cube (length $\sqrt{2}$): each of the 8 unit cubes contributes 8 triangles, one at each corner, for $8 \cdot 8 = 64.$

Face diagonals of the $2 \times 2 \times 2$ cube (length $2\sqrt{2}$): the three faces meeting at a vertex form one triangle, giving 8 triangles.

Edge-midpoint segments (length $\sqrt{6},$ joining midpoints of two edges): each of the 12 edge midpoints is a vertex of two such triangles, for $\frac{12 \cdot 2}{3} = 8.$

The total is $64 + 8 + 8 = 80.$

Thus, the correct answer is **C**.

Problems: https://live.poshenloh.com/past-contests/amc12/2005A