# 2004 AMC 12B Solutions

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1. At each basketball practice last week, Jenny made twice as many free throws as she made at the previous practice. At her fifth practice she made $48$ free throws. How many free throws did she make at the first practice?

   A $3$

   B $6$

   C $9$

   D $12$

   E $15$

   ## Solution:

   Each practice she made twice the previous, so going backward we halve. From the fifth practice at $48$, the earlier practices had $24, 12, 6,$ and $3$ free throws.

   Thus, the correct answer is **A**.

**2.** In the expression $c \cdot a^b - d$, the values of $a, b, c,$ and $d$ are $0, 1, 2,$ and $3,$ although not necessarily in that order. What is the maximum possible value of the result?

- A $5$
- B $6$
- C $8$
- D $9$
- E $10$

## Solution:

To maximize, set $d = 0$. With $a, b, c$ taking $1, 2, 3,$ the term $c \cdot a^b$ is largest when $c = 1$ and $a^b = 3^2 = 9$. This gives $1 \cdot 9 - 0 = 9,$ which beats $2^3 = 8$ and the other assignments.

Thus, the correct answer is **D**.

**3.** If $x$ and $y$ are positive integers for which $2^x 3^y = 1296$, what is the value of $x + y$?

- A $8$
- B $9$
- C $10$
- D $11$
- E $12$

## Solution:

Factoring, $1296 = 6^4 = 2^4 \cdot 3^4$. Matching exponents gives $x = 4$ and $y = 4$, so $x + y = 8$.

Thus, the correct answer is **A**.

**4.** An integer $x$, with $10 \leq x \leq 99$, is to be chosen. If all choices are equally likely, what is the probability that at least one digit of $x$ is a $7$?

A $\frac{1}{9}$

B $\frac{1}{5}$

C $\frac{19}{90}$

D $\frac{2}{9}$

E $\frac{1}{3}$

## Solution:

There are $90$ integers from $10$ to $99$. Ten have a units digit $7$, and nine have a tens digit $7$. Since $77$ is counted twice, there are $10 + 9 - 1 = 18$ with at least one $7$. The probability is $\frac{18}{90} = \frac{1}{5}$.

Thus, the correct answer is **B**.

5. On a trip from the United States to Canada, Isabella took $d$ U.S. dollars. At the border she exchanged them all, receiving $10$ Canadian dollars for every $7$ U.S. dollars. After spending $60$ Canadian dollars, she had $d$ Canadian dollars left. What is the sum of the digits of $d$?

- A $5$
- B $6$
- C $7$
- D $8$
- E $9$

## Solution:

Exchanging gives $\frac{10d}{7}$ Canadian dollars. After spending $60$, she has $\frac{10d}{7} - 60 = d.$ Then $\frac{3d}{7} = 60$, so $d = 140$. The sum of its digits is $1 + 4 + 0 = 5.$

Thus, the correct answer is **A**.

**6.** Minneapolis-St. Paul International Airport is $8$ miles southwest of downtown St. Paul and $10$ miles southeast of downtown Minneapolis. Which of the following is closest to the number of miles between downtown St. Paul and downtown Minneapolis?

A $13$

B $14$

C $15$

D $16$

E $17$

## Solution:

Southwest and southeast are perpendicular, so the airport sits at the right angle of a right triangle with legs $8$ and $10$. The distance between downtowns is $\sqrt{10^2 + 8^2} = \sqrt{164} \approx 12.8$, closest to $13$.

Thus, the correct answer is **A**.

**7.** A square has sides of length $10$, and a circle centered at one of its vertices has radius $10$. What is the area of the union of the regions enclosed by the square and the circle?

A $200 + 25\pi$

B $100 + 75\pi$

C $75 + 100\pi$

D $100 + 100\pi$

E $100 + 125\pi$

## Solution:

The square has area $100$ and the circle has area $100\pi$. Their overlap is the quarter of the circle lying inside the square, with area $25\pi$. The union is $100 + 100\pi - 25\pi = 100 + 75\pi$.

Thus, the correct answer is **B**.

**8.** A grocer makes a display of cans in which the top row has one can and each lower row has two more cans than the row above it. If the display contains $100$ cans, how many rows does it contain?

A $5$

B $8$

C $9$

D $10$

E $11$

## Solution:

The rows contain $1, 3, 5, \ldots, (2n-1)$ cans, and the sum of the first $n$ odd numbers is $n^2$. Setting $n^2 = 100$ gives $n = 10$.

Thus, the correct answer is **D**.

**9.** The point $(-3, 2)$ is rotated $90^\circ$ clockwise around the origin to point $B$. Point $B$ is then reflected in the line $y = x$ to point $C$. What are the coordinates of $C$?

A $(-3, -2)$

B $(-2, -3)$

C $(2, -3)$

D $(2, 3)$

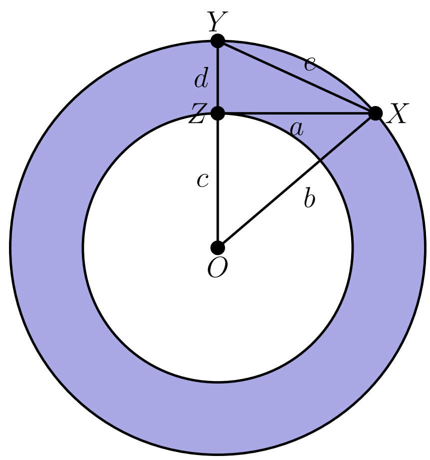
E $(3, 2)$

## Solution:

Rotating $(-3, 2)$ by $90^\circ$ clockwise sends $(x, y) \to (y, -x)$, giving $B = (2, 3)$. Reflecting in $y = x$ swaps coordinates, giving $C = (3, 2)$.

Thus, the correct answer is **E**.

**10.** An *annulus* is the region between two concentric circles. The concentric circles in the figure have radii $b$ and $c$, with $b > c$. Let $\overline{OX}$ be a radius of the larger circle, let $\overline{XZ}$ be tangent to the smaller circle at $Z$, and let $\overline{OY}$ be the radius of the larger circle that contains $Z$. Let $a = XZ, d = YZ$, and $e = XY$. What is the area of the annulus?




A $\pi a^2$

B $\pi b^2$

C $\pi c^2$

D $\pi d^2$

E $\pi e^2$

## Solution:

The annulus area is $\pi b^2 - \pi c^2$. Because $\overline{XZ}$ is tangent to the smaller circle at $Z$, it is perpendicular to radius $\overline{OZ}$, so $\triangle OZX$ is right-angled at $Z$. Then $b^2 = c^2 + a^2$, giving $b^2 - c^2 = a^2$. The area is $\pi a^2$.

Thus, the correct answer is **A**.

**11.** All the students in an algebra class took a $100$-point test. Five students scored $100,$ each student scored at least $60,$ and the mean score was $76.$ What is the smallest possible number of students in the class?

A $10$

B $11$

C $12$

D $13$

E $14$

## Solution:

Each score of $100$ is $24$ above the mean, so the five contribute $120$ points above $76.$ These must be balanced by points below the mean, and each remaining student is at most $76 - 60 = 16$ below. So at least $\frac{120}{16} = 7.5,$ hence $8$ more students are needed, for a total of $13.$ Five $100$s and eight $61$s achieve this.

Thus, the correct answer is **D**.

**12.** In the sequence $2001, 2002, 2003, \ldots$, each term after the third is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the fourth term is $2001 + 2002 - 2003 = 2000$. What is the $2004$th term in this sequence?

- A $-2004$
- B $-2$
- C $0$
- D $4003$
- E $6007$

## Solution:

The rule gives $2001, 2002, 2003, 2000, 2005, 1998, \ldots$ The even-indexed terms are $2002, 2000, 1998, \ldots$, decreasing by $2$. The $2004$th term is the $1002$nd of these: $2002 + 1001(-2) = 0$.

Thus, the correct answer is **C**.

**13.** If $f(x) = ax + b$ and $f^{-1}(x) = bx + a$ with $a$ and $b$ real, what is the value of $a + b$?

A $-2$

B $-1$

C $0$

D $1$

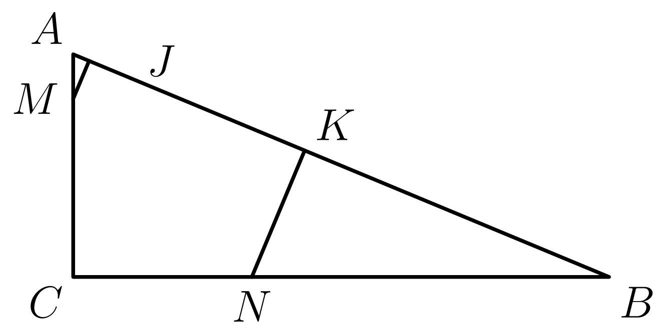
E $2$

## Solution:

Since $f(f^{-1}(x)) = x$, we have $a(bx + a) + b = x$. Matching terms gives $ab = 1$ and $a^2 + b = 0$. Then $b = 1/a$ and $a^2 + 1/a = 0$, so $a^3 = -1$, giving $a = -1$ and $b = -1$. Thus $a + b = -2$.

Thus, the correct answer is **A**.

**14.** In $\triangle ABC$, $AB = 13$, $AC = 5$ and $BC = 12$. Points $M$ and $N$ lie on $\overline{AC}$ and $\overline{BC}$, respectively, with $CM = CN = 4$. Points $J$ and $K$ are on $\overline{AB}$ so that $\overline{MJ}$ and $\overline{NK}$ are perpendicular to $\overline{AB}$. What is the area of pentagon $CMJKN$?




A $15$

B $\frac{81}{5}$

C $\frac{205}{12}$

D $\frac{240}{13}$

E $20$

## Solution:

Since $5^2 + 12^2 = 13^2$, $\triangle ABC$ is right-angled at $C$ with area $\frac{1}{2}(5)(12) = 30$. The small right triangles $\triangle AMJ$ and $\triangle NBK$ are each similar to $\triangle ABC$, with hypotenuses $AM = 5 - 4 = 1$ and $BN = 12 - 4 = 8$. Their areas are $\left(\frac{1}{13}\right)^2(30)$ and $\left(\frac{8}{13}\right)^2(30)$.

The pentagon is what remains:

$$\left(1 - \frac{1}{169} - \frac{64}{169}\right)(30) = \frac{104}{169}(30) = \frac{240}{13}.$$

Thus, the correct answer is **D**.

**15.** The two digits in Jack's age are the same as the digits in Bill's age, but in reverse order. In five years Jack will be twice as old as Bill will be then. What is the difference in their current ages?

- A $9$
- B $18$
- C $27$
- D $36$
- E $45$

## Solution:

Let Jack be $10x + y$ and Bill be $10y + x$. Then $10x + y + 5 = 2(10y + x + 5)$, so $8x = 19y + 5$. Testing digits, only $y = 1, x = 3$ works, so Jack is $31$ and Bill is $13$. The difference is $31 - 13 = 18$.

Thus, the correct answer is **B**.

**16.** A function $f$ is defined by $f(z) = i\overline{z}$, where $i = \sqrt{-1}$ and $\overline{z}$ is the complex conjugate of $z$. How many values of $z$ satisfy both $|z| = 5$ and $f(z) = z$?

- A $0$
- B $1$
- C $2$
- D $4$
- E $8$

## Solution:

Writing $z = x + iy$, we get $f(z) = i(x - iy) = y + ix$. Setting $f(z) = z$ gives $y = x$, which is a line through the origin. The condition $|z| = 5$ is a circle, and a line through the center meets the circle in $2$ points.

Thus, the correct answer is **C**.

**17.** For some real numbers $a$ and $b$, the equation

$$8x^3 + 4ax^2 + 2bx + a = 0$$

has three distinct positive roots. If the sum of the base-$2$ logarithms of the roots is $5$, what is the value of $a$?

A $-256$

B $-64$

C $-8$

D $64$

E $256$

## Solution:

The sum of the base-$2$ logarithms is $\log_2(r_1r_2r_3) = 5$, so $r_1r_2r_3 = 2^5 = 32$. By Vieta's formulas on $8x^3 + 4ax^2 + 2bx + a$, the product of the roots is $-\frac{a}{8}$. Thus $-\frac{a}{8} = 32$, giving $a = -256$.

Thus, the correct answer is **A**.

**18.** Points $A$ and $B$ are on the parabola $y = 4x^2 + 7x - 1$, and the origin is the midpoint of $\overline{AB}$. What is the length of $AB$?

A $2\sqrt{5}$

B $5 + \frac{\sqrt{2}}{2}$

C $5 + \sqrt{2}$

D $7$

E $5\sqrt{2}$

## Solution:

Let $B = (a, b)$ and $A = (-a, -b)$. Then $4a^2 + 7a - 1 = b$ and $4a^2 - 7a - 1 = -b$. Subtracting gives $14a = 2b$, so $b = 7a$. Then $4a^2 + 7a - 1 = 7a$ gives $a^2 = \frac{1}{4}$, and $b^2 = 49a^2 = \frac{49}{4}$. So $AB = 2\sqrt{a^2 + b^2} = 2\sqrt{\frac{50}{4}} = 5\sqrt{2}$.

Thus, the correct answer is **E**.

**19.** A truncated cone has horizontal bases with radii $18$ and $2$. A sphere is tangent to the top, bottom, and lateral surface of the truncated cone. What is the radius of the sphere?

A $6$

B $4\sqrt{5}$

C $9$

D $10$

E $6\sqrt{3}$

## Solution:

The axial cross-section is a trapezoid $ABCD$ with parallel sides $2$ and $18$ and an inscribed circle (a great circle of the sphere). By equal tangent lengths from $B$ and $C$, the slant side $BC = 18 + 2 = 20$. Dropping a perpendicular from $C$ to the bottom base gives a right triangle with horizontal leg $18 - 2 = 16$, so the height is $\sqrt{20^2 - 16^2} = 12$. The sphere's radius is half the height, $6$.

Thus, the correct answer is **A**.

**20.** Each face of a cube is painted either red or blue, each with probability $\frac{1}{2}$. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

- A $\frac{1}{4}$
- B $\frac{5}{16}$
- C $\frac{3}{8}$
- D $\frac{7}{16}$
- E $\frac{1}{2}$

## Solution:

There are $2^6 = 64$ colorings. A suitable orientation exists when all six faces are one color ($2$ ways), exactly five faces are one color ($2 \cdot 6 = 12$ ways), or four faces are one color with the other color on a pair of opposite faces ($2 \cdot 3 = 6$ ways). That is $2 + 12 + 6 = 20$ favorable colorings, so the probability is $\frac{20}{64} = \frac{5}{16}$.

Thus, the correct answer is **B**.

**21.** The graph of $2x^2 + xy + 3y^2 - 11x - 20y + 40 = 0$ is an ellipse in the first quadrant of the $xy$-plane. Let $a$ and $b$ be the maximum and minimum values of $\frac{y}{x}$ over all points $(x, y)$ on the ellipse. What is the value of $a + b$?

A $3$

B $\sqrt{10}$

C $\frac{7}{2}$

D $\frac{9}{2}$

E $2\sqrt{14}$

## Solution:

The slopes $a$ and $b$ are the values of $m$ for which $y = mx$ meets the ellipse in exactly one point. Substituting gives

$$(3m^2 + m + 2)x^2 - (20m + 11)x + 40 = 0.$$

Setting its discriminant to zero yields $-80m^2 + 280m - 199 = 0$. By Vieta's formulas, $a + b = \frac{280}{80} = \frac{7}{2}$.

Thus, the correct answer is **C**.

**22.** The square

| | | |
|---|---|---|
| $50$ | $b$ | $c$ |
| $d$ | $e$ | $f$ |
| $g$ | $h$ | $2$ |

is a multiplicative magic square. That is, the product of the numbers in each row, column, and diagonal is the same. If all the entries are positive integers, what is the sum of the possible values of $g$?

A $10$

B $25$

C $35$

D $62$

E $136$

## Solution:

From the equal row, column, and diagonal products, every entry can be written in terms of $b: h=\frac{100}{b}, g=\frac{100}{c}, f=\frac{100}{d}$. Comparing rows and columns gives $c=\frac{20}{b}$ and $d=\frac{4}{b}$, hence $g=5b$ and $e=10$.

All entries are positive integers exactly when $b=1,2,$ or $4,$ giving $g=5,10,20.$ Their sum is $35.$

Thus, the correct answer is **C**.

**23.** The polynomial $x^3 - 2004x^2 + mx + n$ has integer coefficients and three distinct positive zeros. Exactly one of these is an integer, and it is the sum of the other two. How many values of $n$ are possible?

A $250{,}000$

B $250{,}250$

C $250{,}500$

D $250{,}750$

E $251{,}000$

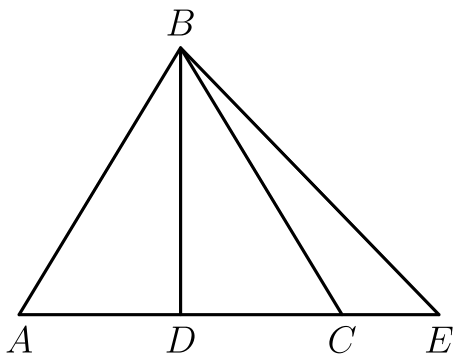
## Solution:

Let the integer zero be $a$. The other two zeros are irrational conjugates $\frac{a}{2} \pm r$, whose sum $a$ equals the integer zero. Vieta's formula on the $x^2$ coefficient gives $a + a = 2004$, so $a = 1002$ and the conjugate pair is $501 \pm r$.

The coefficients are integers exactly when $r^2$ is a positive integer, and the zeros are positive and distinct when $1 \le r^2 \le 501^2 - 1 = 251{,}000$. Since $r$ cannot be an integer, we exclude the $500$ perfect-square values $r^2 = 1^2, \ldots, 500^2$, leaving $251{,}000 - 500 = 250{,}500$ values of $n$.

Thus, the correct answer is **C**.

**24.** In $\triangle ABC, AB = BC$, and $\overline{BD}$ is an altitude. Point $E$ is on the extension of $\overline{AC}$ such that $BE = 10$. The values of $\tan\angle CBE, \tan\angle DBE$, and $\tan\angle ABE$ form a geometric progression, and the values of $\cot\angle DBE, \cot\angle CBE, \cot\angle DBC$ form an arithmetic progression. What is the area of $\triangle ABC$?




A $16$

B $\frac{50}{3}$

C $10\sqrt{3}$

D $8\sqrt{5}$

E $18$

## Solution:

Let $\angle DBE = \alpha$ and $\angle DBC = \beta$. Since $\overline{BD}$ is the altitude of the isosceles triangle, $\angle CBE = \alpha - \beta$ and $\angle ABE = \alpha + \beta$. The geometric progression gives $\tan(\alpha - \beta)\tan(\alpha+\beta) = \tan^2\alpha$, which simplifies to $\tan^2\beta(\tan^4\alpha - 1) = 0$, so $\tan\alpha = 1$ and $\alpha = 45°$.

Writing $DC = a$ and $BD = b$, the arithmetic progression $\cot\angle DBE, \cot\angle CBE, \cot\angle DBC$ becomes $1, \dfrac{b+a}{b-a}, \dfrac{b}{a}$, forcing $b = 3a$. With $BE = 10$ and $\angle DBE = 45°$, we get $b = \dfrac{BE}{\sqrt{2}} = 5\sqrt{2}$, so $a = \dfrac{5\sqrt{2}}{3}$.

The area of $\triangle ABC$ is $\frac{1}{2}(AC)(BD) = ab = 5\sqrt{2}\cdot\dfrac{5\sqrt{2}}{3} = \dfrac{50}{3}$.

Thus, the correct answer is **B**.

**25.** Given that $2^{2004}$ is a $604$-digit number whose first digit is $1$, how many elements of the set $S = \{2^0, 2^1, 2^2, \ldots, 2^{2003}\}$ have a first digit of $4$?

A $194$

B $195$

C $196$

D $197$

E $198$

## Solution:

The smallest power of $2$ with any given digit-count has leading digit $1$. Since $2^{2004}$ has $604$ digits, there are $603$ elements of $S$ with leading digit $1$.

Whenever $2^k$ leads with $1$, $2^{k+1}$ leads with $2$ or $3$, and $2^{k+2}$ leads with $4, 5, 6,$ or $7$. So $603$ elements lead with $2$ or $3$, $603$ lead with $4$ through $7$, and $2004 - 3(603) = 195$ lead with $8$ or $9$.

Finally, $2^k$ leads with $8$ or $9$ exactly when $2^{k-1}$ leads with $4$, so there are $195$ elements with first digit $4$.

Thus, the correct answer is **B**.

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